The Burden of Past Promises*

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Abstract

A manager and an employee are in an infinitely repeated relationship. Effort and profits are observable but not contractible. The opportunity costs of bonus payments are privately observed by the manager. The optimal relational contract trades off the benefits of adapting bonus payments to their opportunity costs with the costs of conflict that arise when bonus payments are contingent on those opportunity costs. Consecutive periods with high opportunity costs lead to a gradual reduction in effort and eventually induce the manager to promise a guaranteed bonus that will be paid even if opportunity costs are high. Such conflicts end with a single period with low opportunity costs during which the manager pays a large bonus and the employee agrees to return a pre-conflict effort level. The optimal relational contract therefore gives rise to asymmetric and infinitely repeated cycles during which profits and effort decline gradually and recover instantaneously. If the firm is liquidity constrained, recoveries becomes sluggish and the employee may increase effort during a conflict.

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1 Introduction

Managers often rely on informal promises rather than formal contracts to motivate their workers. One problem with such "relational contracts" is that employees may not be able to observe whether managers are complying with their promises. This will be the case in particular if the managers' promises are contingent on information that only they observe. The employees' inability to observe compliance then gives rise to conflicts during which employees punish managers by withholding effort, producing low quality goods, departing for other firms and the like. Evidence from recent labor disputes, and case studies about firms that rely on relational contracts, suggest that such worker punishments can impose significant costs on firms (Krueger and Mas 2004, Mas 2008, and the case studies discussed below). The prevalence of relational contracts, their apparent susceptibility to conflicts, and the potential cost of these conflicts raise the question of what managers can do to manage such conflicts and thus make relational contracts more efficient. The aim of this paper is to provide an answer to this question.

In the specific setting we examine a manager and an employee are in an infinitely repeated relationship. The employee's effort and the firm's profits are observable but not contractible. To motivate the employee, the manager must therefore promise him a bonus. The crucial feature of the game is that the firm's opportunity costs of paying the employee can change every period and are privately observed by the manager. Specifically, the firm's opportunity costs may either be high − for instance because of an exceptional investment opportunity, the need to cover losses in another part of the firm or the like − or they may be low. Suppose now that the manager promises the employee a bonus for providing the desired effort level but makes this promise contingent on the firm's opportunity costs being low. If the manager then ends up not paying the bonus, the employee will not be able to observe her motives. Is the manager not paying the bonus because it is more efficient to use resources otherwise, as she claims? Or is she simply making up an excuse to extract some of the employee's rents? Since the employee cannot observe the manager's motives, he must then punish her whenever she does not pay a bonus.

What can the manager do to mitigate this punishment? One option would be for the manager to limit herself to "clear" promises that are not contingent on her private information about the firm's opportunity costs. While such clear promises would allow the manager to avoid conflicts all together, however, they would force her to pay bonuses even when opportunity costs are high. The manager therefore has to trade-off the benefits of flexibility and the costs of conflicts. How does this trade-off evolve over time and to what extent does it depend on the history of the relationship? The optimal relational contract that we derive in this paper answers this question. Before describing it, however, it is useful to discuss two examples that illustrate key features of our set up and of the optimal relational contract.

The first example is Lincoln Electric in the early 1990s. At the time Lincoln Electric was a leading manufacturer of welding machines that was well-known for its unusual incentive structure. One feature of this structure was management's promise to share a significant fraction of profits with its factory workers. In 1992 Lincoln's U.S. business had generated a significant profit and, as a result, its U.S. workers expected to be paid their bonus. Mounting losses in its recently acquired foreign operations, however, more than wiped out U.S. profits. This presented CEO Donald Hastings with a dilemma: *"Our 3,000 U.S. workers would expect to receive, as a group, more than $50 million. If we were in default, we might not be able to pay them. But if we didn't pay the bonus, the whole company might unravel"* (Hastings 1999, p.4). To preserve the relational contract with its U.S. workers, and thus prevent the company from unraveling, management decided to borrow $52.1 million and pay the bonus.

Why would management have to take the seemingly inefficient step of borrowing money to pay the bonus? After all, the bonus was explicitly a "cash-sharing bonus" and U.S. workers had a long history of accepting fluctuations in the bonus in response to fluctuations in U.S. profits. The reason, it seems, was that U.S. workers were unable to observe foreign losses. They therefore could not verify whether U.S. profits really were needed to cover foreign losses or whether money was being hidden from them. This explains why Hastings also "[...] *instituted a financial education program so that employees would understand that no money was being hidden from them* [...]" (Hastings 1999, p.8). At Lincoln Electric, therefore, management's private information about the opportunity costs of bonus payments forced them to trade-off the benefits of flexibility with the costs of conflict.

The second example also took place in the early 1990s but it did so on trading floor of an investment bank rather than factory floor of a manufacturing firm.[4] Specifically, in 1991 management at Credit Suisse made bonus payments to the traders at their First Boston subsidiary that were below those at competing firms. The traders interpreted the low bonus payments as management going back on a previous promise of higher pay. As a result they demanded a larger bonus even if that meant that the firm had to dip into its capital. Management, however, stood firm, arguing that the low bonus payments were justified because of the need to "*build capital*," especially since Credit Suisse had to inject $300m into First Boston the previous year to cover trading losses. To reassure the traders management then "*promised that 1992 would be different – that salaries and*

[4]This example and the following quotes are taken from "Taking the Dare," The New Yorker, 1993.

bonuses would again be competitive." Traders were forthcoming in expressing their disappointment but disruptions were limited.

Things were different, however, the following year when management once again authorized bonus payments that were below the traders' expectations. This time *"many traders seemed to drag their heels, further depressing the firm's earnings"* and *"defections* [...] *increased as soon as First Boston actually began paying bonuses, on March 1st* [1993]*."* Given the severity of the traders' response, management could no longer rely on simply promising more pay next year. Instead:

"*Morale-boosting sessions were hastily organized* [...]. *Some of the investment bankers* [...] *were able to negotiate guaranteed pay raises this year – in some cases of as much as a hundred per cent. Some are waiting to see if such pledges are honored; others acknowledge that the very need for arrangements like these suggests that First Boston is a long way from restoring its culture of trust.*"

Initially, therefore, management chose to adapt bonus payments to their opportunity costs and was willing to accept the traders' response, which turned out to be limited. After two consecutive years of low bonus payments, however, the traders' response was sufficiently severe for management to commit to "guaranteed pay raises." By doing so they forwent the ability to adapt pay to any contingencies without blatantly breaking their promise.

The gradual nature of the traders' punishment, and the eventual need for guaranteed bonus payments, are also features of the optimal relational contract in our setting. To sketch the optimal relational contract, suppose that last period the firm's opportunity costs of paying a bonus were low but that the firm now experiences a number of consecutive periods with high opportunity costs. In this case, how will the manager's promises and the employee's effort evolve over time?

The manager will start out in the current period by promising the employee a bonus if opportunity costs are low but none if they are high. Once the manager finds out that opportunity costs are indeed high she will not pay a bonus and time will move on to the next period.

Next period the manager will then promise the employee a larger bonus than she did in the previous period but once again she will make the bonus contingent on opportunity costs being low. The employee will respond to the failure to pay a bonus in the previous period by reducing effort but this effort reduction will be limited because of the prospect of a larger bonus in case opportunity costs turn out to be low.

As the firm experiences additional periods with high opportunity costs, the manager will promise the employee larger and larger bonuses but she will continue to make these promises contingent on opportunity costs being low. At the same time the employee will respond to the manager's repeated failure to pay a bonus by reducing his effort more and more.

Because of decreasing returns to effort, these effort reductions will become increasingly costly for the firm. Eventually, therefore, the manager will limit further reductions in effort by promising to pay a bonus even if opportunity costs are high. Such a "guaranteed bonus" is therefore symptomatic for a severe crisis and a significant loss of trust, as suggested in the above quote.

If the firm experiences even more periods with high opportunity costs, the manager will initially increase the guaranteed bonus. Eventually, however, the need for the guaranteed bonus to be credible will force the manager to reduce its amount. While the amount of the guaranteed bonus changes over time, it will never be as large as the bonus that the manager promises to pay in case opportunity costs are low. In other words, the manager will never make perfectly clear promises that are entirely independent of the firm's opportunity costs. The reason why the manager never makes entirely clear promises is that she is particularly tempted to renege on her promises when opportunity costs are high. To be credible, a clear promise would therefore have to be quite small. As a result, the manager will always be better off promising a larger bonus if opportunity costs are low than if they are high, even though this difference in promised bonuses necessarily leads to conflicts.

After sufficiently many consecutive periods with high opportunity costs the firm's performance will bottom out. At this point additional periods with high opportunity costs will no longer reduce the employee's effort or change the manager's promises. Even at during these periods, however, the employee will provide strictly positive effort. In our setting, therefore, the employment relationship will never terminate.

What does it take for a conflict to end? No matter how many consecutive periods with high opportunity costs $-$ and thus no or low bonus payments $-$ the firm experiences, it always takes only a single period with low opportunity costs for the conflict to end. Essentially, whenever opportunity costs are low, the manager repays her debts to the employee with a single large bonus. In response the employee agrees to let bygones be bygones and returns to his high, pre-conflict effort level. Overall then the optimal relational contract gives rise to asymmetric cycles during which expected profits and effort decline gradually but recover instantaneously. These cycles continue forever and never end.

The prediction that the manager loses the employee's trust gradually but can restore it instantaneously may be somewhat counter-intuitive. There are two key reasons for this result: first, there is never any uncertainty about the manager's type and, second, she always has access to enough liquidity to repay all of her debts with a single bonus payment. In our main extension we relax this second assumption and instead allow for the bonus payment in any one period to be constrained by the surplus generated in that period. If a conflict is sufficiently severe, it then takes a number of consecutive periods with low opportunity costs for the manager to repay her debts and for the employee's effort to return to its pre-conflict level. Liquidity constraints therefore slow down recoveries.

Another effect of liquidity constraints is to change the dynamics of the employee's punishment. Specifically, the employee may now actually respond to a conflict by increasing effort, at least temporarily. Why would the employee increase effort in response to the manager's failure to pay him a bonus? The reason is that the employee knows that the more effort he provides, the more surplus will be generated, and thus the larger the bonus the manager will be able to pay in case opportunity costs turn out to be low. This reasoning is reflected in the Lincoln Electric example that we started to describe above. In particular, in early 1993 – only a few months after Lincoln Electric had taken on debt to pay the bonus – management realized that European losses would once again wipe out U.S. profits. In the following quote Lincoln Electric CEO Don Hastings describes how management responded to this challenge.

"*So rather than downsize, we turned to our U.S. employees for help. I presented a 21-point plan to the board that called for our U.S. factories to boost production dramatically* [...]*. 'We blew it,' I said* [to the U.S. employees]*. 'Now we need you to bail the company out. If we violate the covenants, banks won't lend us money. And if they don't lend us money, there will be no bonus in December.' Thanks to the Herculean effort in the factories and in the field, we were able to increase revenues and profits enough in the United States to avoid violating our loan covenants.* [...] *On December 4, 1993, we paid a gross bonus of $55.3 million with borrowed money.*"

In line with the prediction that we sketched above, therefore, Lincoln Electric's U.S. employees increased their efforts to relax the firm's liquidity constraints which, in turn, allowed management to pay the bonus.

2 Literature Review

This paper contributes to several strands of literature. Since Bull (1987), there is a prominent line of research on relational contracts. One general lesson from this literature is that the relational contract is sustainable so long as the sum of the reneging temptations do not exceed the total future surplus of the relationship; see for example MacLeod and Malcomson (1989) in the case of perfect information, Baker, Gibbons, and Murphy (1994, 2002) in cases when explicit contracts or ownership structure can be used; Levin (2003) in the case of imperfect public monitoring, Levin

(2002) in the case of multiple agents, and Rayo (2007) when multiple agents, explicit contract, and ownership structure are all present. In these models, the optimal relational contracts can all be implemented by a sequence of stationary contracts so that past events along the equilibrium do not affect the relationship in the future. In contrast, this paper focuses on how past events affect the future of the relationship, and in particular, how players adjust their actions to manage the firm's reneging temptations when the future surplus of the relationship fluctuates.

The dynamics along the equilibrium play relates this model to a recent and growing literature on relational contracts with equilibrium dynamics. Thomas and Worrall (2010) study a partnership game with perfect information and two-sided limited liability and show that the relationship becomes more efficient over time as the division of future rents becomes more equal. Chassang (2010) found that private information prevents the relationship from efficient exploration of new production opportunities and the relationship may settle in different long run equilibrium. Fong and Li (2010) derive implications on how job security, pay level, and the sensitivity of pay to performance change over time when the worker has moral hazard and is protected by limited liability constraint. Padro i Miguel &Yared (2010) show that costly intervention from the principal is inevitable when the agent has asymmetric information and generate predictions on the likelihood, duration, and intensity of intervention. One distinctive feature of our equilibrium dynamic is that the relationship displays sluggish decay and instantaneous recovery: the relationship cycles over time and does not converge to a steady state. The difference in long-run dynamics is in part driven by the source of asymmetric information: all the models above rely on hidden action and we depend on hidden information. Halac (forthcoming), Watson (1999, 2002), and Yang (2009) study relational contracts in which the type of the agents are their private information. In these models, the type of the agent is fixed, and dynamics arises because the principal update his belief of the type of agent based on the past history. The equilibrium play in these models converges to a long-run steady state. In our model, the type of the firm is independently distributed over time, and past history does not contain information about type of the firm in the future.

Our model is closer to the literature of dynamic games with hidden information; see for example Abdulkadiroglu and Bagwell (2010), Athey and Bagwell (2001, 2008), Athey, Bagwell and Sanchirico (2004), Hauser and Hopenhyan (2008), and Mobius (2001). We add to this literature in two aspects. First, this literature has typically studied relationships with symmetrical players and multi-sided private information, and we model an asymmetric relationship with one-sided private information. One qualitative difference is that first-best is no longer obtainable in a discrete time setting even if the players are sufficiently patient. Second, this literature has mostly focused on settings in which transfer is not possible, and we allow for free transfers in non-shock states and costly ones in shock states. If transfer is free in all states, Levin (2003) shows that the optimal relational contract with hidden information is stationary and constrained efficient for sufficiently patient players. In our model, we show that a small probability in costly transfer can greatly change the structure of the relationship by generating inefficiency in every period.

To the extent that efficiency calls for the firm to pay only in non-shock states, our model has a flavor of risk-sharing. Kocherlakota (1996) studies efficient risk sharing without commitment between two risk-averse agents when information is public. Hertel (2004) examines the case with two-sided asymmetric information without commitment. Thomas and Worrall (1990) study an one-sided asymmetric information problem but assumes away the commitment issue. Our model has one-sided asymmetric information and adds to the literature in two other aspects. First, this literature has traditionally focused more on long-run outcomes and we supplement it with more detailed description of how the relationship evolves. Second, the literature has typically assumed that the agent's endowments are exogenously given whereas we have focused on a case in which the sizes of the pies in the future depend on they were divided in the past.

3 The Model

A firm and a worker are in an infinitely repeated relationship. Time is discrete and denoted by $t = \{1, 2, ..., \infty\}$.

At the beginning of any period t the firm makes the worker a non-binding offer. For reasons that will become clear shortly, the offer consists of two bonus payments $b_{n,t}$ and $b_{s,t}$, where the subscripts n and s stand for "no shock" and "shock." After the firm makes the offer, the worker either accepts the offer or rejects it. We denote the worker's decision by d_t, where $d_t = 0$ if he rejects the offer and $d_t = 1$ if he accepts it. If the worker rejects the offer, the firm and the worker realize their per-period outside options $\underline{\pi} > 0$ and $\underline{u} > 0$ and time moves on to period $t + 1$.

If, instead, the worker accepts the firm's offer, he next decides on his effort level $e_t \geq 0$. Effort is costly to the worker and we denote his effort costs by $c(e_t)$. We assume that $c(0) = c'(0) = 0$ and that for all $e_t \geq 0$, $c'(e_t) \geq 0$, $c''(e_t) > 0$, and $\lim_{e_t \to \infty} c'(e_t) = \infty$. After the worker provides effort e_t, the firm realizes output $y(e_t)$. We assume that $y(0) = 0$ and that for all $e_t \geq 0$, $y'(e_t) \geq 0$, and $y''(e_t) < 0$. Effort e_t, effort costs $c(e_t)$, and output $y(e_t)$ are observable by the worker and the firm but they are not contractible.

After the firm receives output $y(e_t)$, it privately observes whether or not it has been hit by a shock. Specifically, the firm privately observes the state of the world $\Theta_t \in \{n, s\}$, where n

and s stand for "no shock" and "shock." The shock state occurs with probability $\theta \in (0,1)$ and the no shock state occurs with probability $(1-\theta)$. The state of the world determines the firm's opportunity cost of paying the worker: if no shock occurs, paying the worker some bonus b costs the firm b; if, instead, a shock does occur, paying the worker b costs the firm $(1+\alpha)\,b$, where $\alpha > 0$. We do not model explicitly why the firm's opportunity costs may be high. As discussed in the Introduction, however, firms do sometimes face high opportunity costs of paying their workers, for instance, because they need to borrow money to make their payments, as in the Lincoln Electric example.

After the firm observes the state of the world, it pays the worker a bonus $b_t \geq 0$. Since the firm's initial offer was non-binding, the bonus payment b_t can be different from the promised bonus payments $b_{n,t}$ and $b_{s,t}$. The bonus payment does, however, have to be positive since the worker is assumed to be liquidity constrained.

Finally, at the end of period t, the firm and the worker observe the realization x_t of a public randomization device. The existence of a public randomization device is a common assumption in the literature on imperfect public monitoring and is typically made to convexify the set of equilibrium payoffs. The timing is summarized in Figure 1.

The firm and the worker are risk neutral and try to maximize their discounted expected profit and payoff streams. The firm and the worker both discount the future at a rate $\delta \in [0,1)$ per period. At the beginning of any period t, the firm's expected profits are given

$$\pi_t = (1-\delta)\sum_{\tau=t}^{\infty}\delta^{\tau-t}\mathrm{E}\left[\underline{\pi} + d_\tau\left[y\left(e_\tau\right) - \left(1+\theta\alpha\right)b_\tau - \underline{\pi}\right]\right]$$

and the worker's expected payoff is given by

$$u_t = (1-\delta)\sum_{\tau=t}^{\infty}\delta^{\tau-t}\mathrm{E}\left[\underline{u} + d_\tau\left(b_\tau - \underline{u} - c\left(e_\tau\right)\right)\right].$$

Note that we multiply the RHS of each expression by $(1-\delta)$ to express profits and payoffs as per period averages.

We follow the literature on imperfect public monitoring and restrict attention to Perfect Public Equilibria in which (i.) the firm and the worker play public strategies and (ii.) for every date and every history, the strategies are a Nash Equilibrium from that point on. Public strategies are strategies in which the players condition their actions only on publicly available information.

Formally, let $h_{t+1} = \{b_{n,\tau},\, b_{s,\tau},\, d_\tau,\, e_\tau,\, b_\tau,\, x_\tau\}_{\tau=1}^{t}$ denote the public history at the beginning of any period $t+1$ and let $H_{t+1} = \{h_{t+1}\}$ denote the set of all possible histories. Note that $H_1 = \Phi$. A public strategy for the firm specifies (i.) for every period and every possible public history, the bonus offer in case a shock occurs and in case it does not occur and (ii.) for every period, every possible public history, every decision by the worker, every effort choice by the worker, and every state, the firm's bonus payments. More precisely, a public strategy for the firm s_F is a sequence of functions $\{B_{s,t},\, B_{n,t},\, B_t\}_{t=1}^{\infty}$, where $B_{s,t} : H_t \rightarrow [0,\infty)$, $B_{n,t} : H_t \rightarrow [0,\infty)$, and $B_t : H_t \cup \{b_{s,t},\, b_{n,t},\, d_t,\, \theta_t\} \rightarrow [0,\infty)$. Similarly, a public strategy for the worker specifies, for every period, every possible public history, and every possible bonus offer, whether he accepts or rejects the firm's offer and how much effort he provides. More precisely, a public strategy for the worker s_W is a sequence of functions $\{D_t,\, E_t\}_{t=1}^{\infty}$, where $D_t : H_t \cup \{b_{s,t},\, b_{n,t}\} \rightarrow \{0,1\}$ and $E_t : H_t \cup \{b_{s,t},\, b_{n,t},\, d_t\} \rightarrow [0,\infty)$.

We denote by $\pi_t\left(s_F, s_W\right)$ and $u_t\left(s_F, s_W\right)$ the firm's expected profits and the worker's expected payoff at the beginning of period t, conditional on the public strategies s_F and s_W. Our aim is to characterize the Perfect Public Equilibria that are Pareto efficient in the first period.

Before we move on to solve this model, it is worth making a few observations. First, note that in contrast to the worker, the firm is never liquidity constrained. In principle, the firm can therefore pay the worker any positive amount, even when its opportunity costs of doing so are high. We relax this assumption in Section 6. Second, we assume throughout the paper that the firm cannot save across periods. We make this assumption for simplicity. Note, however, that as long as the firm is not liquidity constrained, this assumption is innocuous. Finally, note that our model is similar to a standard trust game (references). In line with the literature on trust games, one can interpret the worker's effort as a measure of how much he trusts the firm.

4 Preliminaries

4.1 PPE Payoff Set and Recursive Formulation

In this section, we use the technique developed by Abreu, Pearce, Stacchetti (1990) to characterize the PPE payoff. The basic idea of Abreu, Pearce, Stacchetti (1990) is that a player's total payoff in a repeated game can be decomposed into a current payoff and a future continuation payoff. This transforms an infinitely repeated game into a two stage game. The action of a player maximizes the weighted average of his payoff in these two stages. The key insight of Abreu, Pearce, Stacchetti (1990) is that as long as the continuation payoff lies in the PPE payoff set, and can therefore be supported by some equilibrium play, the player's total payoff can also be supported by a PPE. This allows the PPE payoff set to be characterized recursively.

Denote the PPE payoff set of our game by E and consider a payoff pair $(\pi, u) \in E$ such that the firm and the worker join the relationship in the stage game. To support (π, u) as a PPE payoff pair, one specifies in the stage game the agent's effort level e and the principal's bonus payment b_s and b_n depending on the state of the world. In addition, (π, u) assigns a continuation payoff to each publicly observable outcomes. Let (π_s, u_s) and (π_n, u_n) be continuation payoff pairs along the equilibrium path in the shock and no shock states. The set of necessary and sufficient conditions for (π, u) to be a PPE payoff is such that (a.) the set of actions and continuations payoffs specified are feasible, (b.) the players cannot benefit from deviating to other actions, and (c.) (π, u) is equal to the weighted average of current payoff and future continuation payoff.

Feasibility For the actions to be feasible, the requirement is that the bonuses are non-negative and so is the effort level. Specifically, we need

$$b_s \geq 0, \tag{$\mathrm{Neg_S}$}$$

$$b_n \geq 0, \tag{$\mathrm{Neg_N}$}$$

and

$$e \geq 0. \tag{$\mathrm{Neg_e}$}$$

Recall that the non-negativity constraints are the bonus levels reflect the limited liability of the agent. The constraint on effort is a normalization.

For the continuation payoffs to be feasible, we need that the continuation payoffs are also PPE payoffs. In other words, we need

$$(\pi_s, u_s) \in E \tag{$\mathrm{Self\text{-}Enf_S}$}$$

and

$$(\pi_n, u_n) \in E. \tag{$\mathrm{Self\text{-}Enf_N}$}$$

No Deviation For the players not to deviate, we need to consider two types of deviations: off-schedule and on-schedule. Off-schedule deviations are those that can be publicly observed. If an off-schedule deviation occurs, there is no loss of generality in assuming that the players will permanently break up the relationship (by taking their outside options), as this is the worst possible equilibrium (Abreu 1988). Here, the principal deviates off-schedule when he fails to pay a bonus equalling either b_s or b_n. When this occurs, the principal's continuation payoffs will always be $\underline{\pi}$.

To prevent the principal from off-schedule deviations, it suffices that his loss in future continuation payoff exceeds his maximum possible current gain from deviating. Note that the principal's current gain from deviation is maximized when he pays zero bonus, and this implies that the principal will not deviate off-schedule when the gain from reneging the bonus in its entirety is smaller than the loss in his continuation value. This gives us the inequalities

$$\delta\pi_s - \delta\underline{\pi} \geq (1-\delta)(1+\alpha)\, b_s \tag{RC$_\mathrm{S}$}$$

and

$$\delta\pi_n - \delta\underline{\pi} \geq (1-\delta)\, b_n. \tag{RC$_\mathrm{N}$}$$

The first inequality prevents the principal from off-schedule deviation in the shock state and second one in the no shock state. In both inequalities, the left hand sides are future losses in continuation from off-schedule deviation and the right hand sides are the maximal current gain.

For the agent, he deviates off-schedule when he does not put in effort e. When this occurs, the agent will receive zero bonus and that his continuation payoff will be $\underline{u}$. By deviating away from e, the agent gains most by putting in zero level of effort. Therefore, to prevent the agent from off-schedule deviation, it suffices that the agent's payoff from putting in zero effort and receiving a continuation payoff of $\underline{u}$ is smaller than his payoff on the equilibrium path. In other words,

$$\delta\underline{u} \leq u. \tag{IC$_\mathrm{W}$}$$

In addition to off-schedule, we need to consider on-schedule deviations, which are those privately observed by the player. Since the state of the world is the principal's private information, on-schedule deviations include that the principal paying out b_s in a no-shock state or paying out b_n in a shock state. To prevent the principal from paying out b_s in a no-shock state, we need

$$\delta(\pi_n - \pi_s) \geq (1-\delta)(b_n - b_s). \tag{IC$_\mathrm{N}$}$$

To prevent the principal from paying out b_n in a shock state, we need

$$\delta(\pi_n - \pi_s) \leq (1+\alpha)(1-\delta)(b_n - b_s). \tag{IC$_\mathrm{S}$}$$

Promise-Keeping Finally, the consistency of the PPE payoff decomposition requires that the players' payoffs are equal to the weighted sum of current and future payoffs. Specifically, we have

$$\pi = \theta\left((1-\delta)\left(y(e) - (1+\alpha)b_s\right) + \delta\pi_s\right) + (1-\theta)\left((1-\delta)\left(y(e) - b_n\right) + \delta\pi_n\right) \tag{PK$_\mathrm{F}$}$$

and

$$u = \theta(1-\delta)\, b_s + \theta\delta u_s + (1-\theta)(1-\delta)\, b_n + (1-\theta)\,\delta u_n - (1-\delta)\, c(e)\,. \tag{PK$_\mathrm{W}$}$$

4.2 PPE Payoff Frontier

In general, characterizing the PPE payoff set is complicated because it is a two-dimensional set. In our game, however, E is completely characterized by its payoff frontier, a one-dimensional curve. This simplification results from the fact that taking outside options for all periods is a PPE and that we have public randomization. Define the payoff frontier as

$$u(\pi) = \max\{u' : (\pi, u') \in E\}.$$

The next lemma shows that every payoff pair below the payoff frontier u and above the agent's outside option $\underline{u}$ belongs to the PPE payoff set. In addition, u is concave and at the principal's maximal PPE payoff, the agent's payoff is equal to his outside option.

LEMMA A1. *Let $\overline{\pi}$ be the maximum PPE payoff of the principal. The PPE payoff set E is given by*

$$E = \{(\pi', u') : \pi' \in [\underline{\pi}, \overline{\pi}], u' \in [\underline{u}, u(\pi')]\}.$$

In addition, u is concave, and

$$u(\overline{\pi}) = \underline{u}.$$

To characterize the payoff frontier u, we first show that exists a cutoff value π_0 that divides the frontier into two areas. To the left of π_0, the payoff frontier is sustained by a randomization between the payoffs from outside options $(\underline{\pi}, \underline{u})$ and $(\pi_0, u(\pi_0))$. To the right of π_0, the payoff frontier is sustained by pure strategies such that the principal and the agent join the relationship in the stage game. Note that it is possible that $\pi_0 = \underline{\pi}$. In this case, the entire payoff frontier is sustained by pure strategies.

LEMMA A2. *There exists a π_0 such that for all $\pi \in [\pi_0, \overline{\pi}]$, $u(\pi)$ can be sustained by pure strategy.*

For payoffs larger than π, an important property of the PPE payoff frontier is that it is sequentially optimal. Specifically, for each payoff pair on the frontier, all of the continuation payoffs must also be on the frontier.

LEMMA A3. *Let $(\pi, u(\pi))$ be a PPE payoff sustained by pure strategy in period 1. Let (π_s, u_s) and (π_n, u_n) be the continuation payoffs following the shock and no shock state respectively. Then*

$$u_s = u(\pi_s)$$

and

$$u_n = u(\pi_n).$$

The key reason for the lemma above is that the actions of the agent can be perfectly observed. This implies that any deviation from the agent is publicly known. Along any equilibrium path, no public deviation can occur, and, thus, the agent has not deviated. Therefore, optimal PPE should not destroy surplus along the equilibrium path to punish the agent. In other words, in an optimal PPE the agent's payoff following any equilibrium path always lies on the PPE payoff frontier. This result is common in repeated games in which one party's action is perfectly observed; see for example Levin (2003) and Fong and Li (2010). In general repeated games with imperfect public monitoring, however, payoff can fall below the PPE payoff frontier when no player can guarantee that he has not deviated along the equilibrium path; see for example Green and Porter (1984), Athey and Bagwell (2001).

The previous lemmas establish that, to characterize the payoff frontier to the right of π_0, it suffices to specify effort e, bonus levels b_s, and b_n, and continuation payoffs of the principal π_s and π_n. To simplify our analysis of the payoff frontier, we now proceed to eliminate the redundant constraints associated with the PPE payoffs. We start with the constraints associated with feasibility.

Feasibility By combining $\mathrm{IC_N}$ and $\mathrm{IC_S}$, we have

$$(1+\alpha)(1-\delta)(b_n - b_s) \geq \delta(\pi_n - \pi_s) \geq (1-\delta)(b_n - b_s).$$

This implies as long as these two constraints are satisfied, we must have

$$b_n \geq b_s.$$

Consequently, the non-negativity constraint on b_n is automatically satisfied when the non-negativity constraints are satisfied.

For the continuation payoffs to be feasible, it suffices that the principal's continuation payoffs can be supported as a PPE payoff since the agent's payoff is always on the PPE payoff frontier. In other words, the feasibility constraints of the continuation payoffs are

$$\begin{aligned} \underline{\pi} &\leq \pi_s \leq \overline{\pi}; \\ \underline{\pi} &\leq \pi_n \leq \overline{\pi}. \end{aligned}$$

Note that $\pi_n \geq \pi_s$ because $b_n \geq b_s$. Therefore, the constraints above can be reduced to $\underline{\pi} \leq \pi_s$ and $\pi_n \leq \overline{\pi}$. Note that $\underline{\pi} \leq \pi_s$ is implied by $\mathrm{RC_S}$ and the non-negativity of b_s. Therefore, the only non-redundant constraint here is

$$\pi_n \leq \overline{\pi}. \qquad \text{(Self-enf)}$$

No Deviation Next, we examine the constraints associated with No Deviation. It is clear IC_{W} is automatically satisfied by the definition of the payoff frontier and that $\underline{u} > 0$. To eliminate more constraints, we first establish below the familiar result that IC_{N} is satisfied with equality.

LEMMA A4.

$$\delta\left(\pi_n - \pi_s\right) = (1-\delta)\left(b_n - b_s\right). \tag{IC_{N}}$$

Once IC_{N} is satisfied with equality, we see that IC_{S} is automatically satisfied since $b_n \geq b_s$. In addition, RC_{N} becomes redundant once RC_{S} is satisfied. To see this, note that

$$\begin{aligned}
\delta\pi_n - \delta\underline{\pi} &= \delta\pi_n - \delta\underline{\pi} + (1-\delta)\left(b_n - b_s\right) \\
&\geq (1-\delta)(1+\alpha)\, b_s + (1-\delta)\left(b_n - b_s\right) \\
&\geq (1-\delta)\, b_n,
\end{aligned}$$

where the equality is equality follows from the lemma above, the first inequality uses RC_{S}, and the second inequality uses the non-negativity of b_s.

Promise-Keeping For the promise-keeping constraints, we make the following changes. First, we solve the promise keeping constraint for the principal PK_{F} for the no shock bonus b_n and then substitute into the IC_{N} constraint. Doing so we obtain

$$\pi = (1-\delta)\, y(e) + \delta\pi_s - (1-\delta)(1+\theta\alpha)\, b_s. \tag{IC_{N}}$$

Second, we replace the promise-keeping constraint of the agent with a promise-keeping constraint of the value of the relationship. In particular, by adding up the promise-keeping constraints of the principal and the agent, we obtain

$$\pi + u(\pi) = (1-\delta)\left(y(e) - c(e)\right) + \theta\delta\left(\pi_s + u(\pi_s)\right) + (1-\theta)\,\delta\left(\pi_n + u(\pi_n)\right) - (1-\delta)\,\theta\alpha b_s.$$

The left hand side of the equation is the value of the relationship. The right hand side decomposes the value into the weighted average of those generated by the stage game and those from the future.

By the definition of the payoff frontier, any choice of feasible actions and continuation cannot generate value that exceeds $\pi + u(\pi)$. Therefore, the value of an optimal relational contract must satisfy the following:

$$\pi + u(\pi) = \max_{e, b_s, \pi_s, \pi_n} (1-\delta)\left(y(e) - c(e)\right) + \theta\delta\left(\pi_s + u(\pi_s)\right) + (1-\theta)\,\delta\left(\pi_n + u(\pi_n)\right) - (1-\delta)\,\theta\alpha b_s \tag{1}$$

such that

$$\pi = (1-\delta)\, y(e) + \delta \pi_s - (1-\delta)(1+\theta\alpha)\, b_s. \tag{IC$_\mathrm{N}$}$$

$$\delta \pi_s - \delta \underline{\pi} \geq (1-\delta)(1+\alpha)\, b_s. \tag{RC$_\mathrm{S}$}$$

$$b_s \geq 0; \tag{Neg$_\mathrm{S}$}$$

$$e \geq 0. \tag{Neg$_\mathrm{e}$}$$

$$\pi_n \leq \overline{\pi}. \tag{Self-enf}$$

4.3 Key Features of the Payoff Frontier

In this subsection, we state and discuss some key features of the payoff frontier. Our discussion here focuses on the intuitions. A formal treatment can be found in the appendix. In particular, we show that the payoff frontier is differentiable everywhere.

LEMMA 1. *The slope of the payoff frontier $u(\pi)$ satisfies*

$$\frac{\mathrm{d}u(\pi)}{\mathrm{d}\pi} > -1 \quad \textit{for all } \pi \in [\underline{\pi}, \overline{\pi}].$$

An increase in expected profits by one dollar therefore reduces the worker's payoff by less than a dollar and thus increases joint surplus. Essentially, an increase in expected profits relaxes the IC_N constraint and thus allows for an increase in effort e, or a reduction in the inefficient bonus b_s, or both.

The next lemma provides a sufficient condition under which $\pi_0 = \underline{\pi}$, that is, all payoffs on the payoff frontier can be sustained by pure strategies. Moreover, it shows that when that condition is satisfied, the payoff frontier is everywhere downward sloping and thus maximized at $\underline{\pi}$. To state the next and subsequent lemmas, let the profit level at which the $u(\pi)$ is maximized be denoted by π_m, where the subscript stands for "maximum."

LEMMA 2. *Suppose that the magnitude of the shock α is large relative to its frequency θ, in the sense that*

$$\alpha \geq \frac{\theta}{1-\theta}. \tag{A}$$

Then (i.) all payoffs on the payoff frontier can be sustained by pure strategies; in other words, $\pi_0 = \underline{\pi}$. And (ii.), the payoff frontier is everywhere strictly downward sloping, that is, $u'(\pi) < 0$ for all $\pi \in [\underline{\pi}, \overline{\pi}]$. The payoff frontier is therefore maximized at $\underline{\pi}$, that is, $\pi_m = \underline{\pi}$.

5 Characterizing the Optimal Relational Contract

In this section we characterize the optimal relational contract. We do so first for the case in which Condition A is satisfied. Recall from the previous section that all payoffs on the payoff frontier can then be sustained by pure strategies. The optimal relational contract is therefore fully characterized by the solution to the maximization problem (1). Once we have characterized the optimal relational contract when Condition A is satisfied, we turn to the case in which Condition A is not satisfied. While the optimal relational contracts are very similar in both cases, they are more easily described sequentially.

5.1 The Optimal Relational Contract When Condition A Holds

Consider first a period in which the firm expects to make profits $\pi \in [\underline{\pi}, \overline{\pi}]$. We already observed that the optimal bonus in a shock period $b_s^*(\pi)$ is weakly smaller than the optimal bonus in a no shock period $b_n^*(\pi)$. The firm therefore always has a temptation to claim to have been hit by a shock, even when it has not been. To ensure that the firm does not succumb to this temptation, the optimal relational contract rewards the firm whenever it does not claim to have been hit by a shock. In particular, the next lemma shows that if $\pi < \overline{\pi}$ and the firm pays $b_n^*(\pi)$, its expected profits increase to $\overline{\pi}$ at the beginning of the next period. And if expected profits are already at their upper bound $\pi = \overline{\pi}$, they will stay there.

LEMMA 3. *The optimal continuation profit in a period without a shock is given by*

$$\pi_n^* = \overline{\pi} \quad \text{for all } \pi \in [\underline{\pi}, \overline{\pi}].$$

In addition to getting rewarded whenever it does not claim to have been hit by a shock, the firm also gets punished whenever it does claim to have been hit by a shock. In particular, the next lemma shows that if $\pi > \underline{\pi}$ and the firm only pays $b_s^*(\pi)$, its expected profits decline to $\pi_s^*(\pi) < \pi$ at the beginning of the next period. And if expected profits are already at their lower bound $\pi = \underline{\pi}$, they will stay there.

LEMMA 4. *The the optimal continuation profit in a period with a shock is given by*

$$\pi_s^*(\underline{\pi}) = \underline{\pi} \quad \text{and} \quad \pi_s^*(\pi) < \pi \quad \text{for all} \quad \pi > \underline{\pi}.$$

Since the optimal relational contract uses both carrots and sticks to induce the firm to be truthful, expected profits cycle over time. Consider, for instance, Figure 2 which charts expected profits for an arbitrarily chosen sequence of shock and no shock periods. In the figure red squares indicate periods in which the firm has been hit by a shock and blue dots indicate periods in which the firm has not been hit by a shock. The figure illustrates that consecutive shock periods lead to a gradual reduction in expected profits until, after a sufficiently large number of consecutive shocks, expected profits bottom out at $\underline{\pi}$. At any point, however, it only takes a single no shock period for expected profits to return to their upper bound $\overline{\pi}$. Expected profits therefore cycle over time and they do so asymmetrically, with downturns being gradual and recoveries instantaneous. Moreover, since the relationship never terminates, the cycles never end.

It is intuitive that the firm gets punished whenever it claims to have been hit by a shock and only pays the worker $b_s^*(\pi)$. The next question, however, is how this punishment should be implemented. Since the relationship between the firm and the worker never terminates, punishment never takes the form of an increase in the probability of termination. Instead, the firm gets punished for only paying $b_s^*(\pi)$ by having to reallocate some of its future profits to the worker. This, of course, is in line with our previous observation that the payoff frontier is everywhere downward sloping. A reduction in expected profits π therefore implies an increase in the worker's rents $u(\pi) - \underline{u}$. Essentially, the worker will always accept a relatively small bonus payment $b_s^*(\pi)$ but in return the firm has to promise him higher rents in the future.

The next question then is how the firm should increase the worker's future rents. In principle, there are three ways in which the firm could do so: it could pay the worker a larger bonus in a future no shock period, it could pay him a larger bonus in a future shock period, or it could keep the bonuses constant but allow the worker to provide less effort in some future period. The most efficient way to increase the worker's future rents is of course the first one, that is, to pay the worker a larger bonus in a future no shock period. Since the $\mathrm{IC_N}$ constraint is always binding, however, an increase in a future no shock bonus alone is not incentive compatible. To see this, suppose, that whenever the firm is hit by a shock, it does not pay the worker and the worker does not reduce his effort; instead, the firm simply promises him a larger bonus in the next no shock period. Such a punishment strategy would be efficient but not effective, since the firm could avoid any punishment by simply never paying a bonus. To be credible, the promise to increase the worker's future rents therefore has to involve a reduction in effort or an increase in inefficient bonus payments. Essentially, the firm can always claim to have been hit by a shock and only pay the worker $b_s^*(\pi)$, provided that it promises the worker higher rents in the future. This promise, however, constrains the firm's ability to operate efficiently in the future; in particular, it limits the firm's ability to motivate the worker and may force it to make inefficient bonus payments. These future inefficiencies that are what we call the "burden of past promises."

To understand how the firm increases the worker's future rents, consider first $e^*(\pi)$, the worker's optimal effort in a period in which the firm expects to make profits π. We characterize $e^*(\pi)$ in the next lemma.

LEMMA 5. *Optimal effort $e^*(\pi)$ satisfies*

$$0 < e^*(\pi) < e_{fb} \text{ for all } \pi \in [\underline{\pi}, \overline{\pi}].$$

where e_{fb} is first best effort. Moreover, $e^*(\pi)$ is weakly increasing in π and satisfies

$$e^*(\pi_s^*(\pi)) < e^*(\pi) \quad \text{for all } \pi_s^*(\pi) < \pi.$$

If the firm only pays $b_s^*(\pi)$ this period, and expected profits are not yet at their lower bound $\underline{\pi}$, the worker will therefore provide strictly less effort in the next period. And if the firm only pays $b_s^*(\pi)$ in consecutive periods, effort continues to decline until it bottoms out at $e^*(\underline{\pi})$. The firm's punishment for a small bonus payment is therefore not delayed to some far off, future period. Instead, the punishment starts in the next period and continues until the next time the firm is not hit by a shock and pays the optimal no shock period $b_n^*(\pi)$.

Notice also that minimum effort $e^*(\underline{\pi})$ is strictly positive. The worker therefore always provides some effort, even if the firm has been hit by an arbitrarily large number of consecutive shocks. At the same time, effort always falls short of first best, even if the firm and the worker are very patient. We elaborate on the failure to achieve first best in Section 5.1.

Consider next the optimal bonus $b_s^*(\pi)$ that the firm pays to worker in a period in which it expects to make π and is then hit by a shock. The next lemma provides a necessary and sufficient condition for the firm to never pay a bonus in a shock state.

LEMMA 6. *Let $\underline{e} > 0$ denote the effort level for which $y(\underline{e}) = \underline{\pi}$. Then the optimal bonus in a shock state $b_s^*(\pi)$ is equal to zero for all $\pi \in [\underline{\pi}, \overline{\pi}]$ if and only if*

$$\frac{c'(\underline{e})}{y'(\underline{e})} \geq \frac{1 - \alpha(1-\theta)}{1 + \theta\alpha}. \tag{B}$$

Condition B is more likely to be satisfied the larger the magnitude of the shock α relative to its frequency θ and relative to the firm's outside option $\underline{\pi}$. We therefore have the intuitive result that the firm never pays a bonus in a shock state if the opportunity costs are sufficiently large.

The Optimal Relational Contract Does Not Involve Inefficient Bonus Payments Suppose first that Condition B holds so that the optimal relational contract does not involve inefficient bonus payments. We can then complete the characterization of the optimal relational contract by substituting $\pi_n^*(\pi)$, $\pi_s^*(\pi)$, and $b_s^*(\pi)$ into the IC_N constraint to obtain the optimal bonus in a no shock period.

LEMMA 7. *Suppose that Condition B holds. Then the optimal bonus in a period without a shock is given by*

$$b_n^*(\pi) = \frac{\delta}{1-\delta}\left(\overline{\pi} - \pi_s^*(\pi)\right).$$

If the firm does not pay a bonus this period, and if expected profits are not yet at their lower bound $\underline{\pi}$, the firm therefore has to pay a larger bonus in the next no shock period. And if the firm does not pay a bonus in consecutive periods, the optimal no shock bonus increases until it tops out at its upper bound $b_n^*(\underline{\pi})$. The punishment for not paying a bonus therefore involves both an immediate reduction in effort and an immediate increase in the next no shock bonus. And this punishment continues until the next time the firm is not hit by a shock and pays the optimal no shock bonus. The evolution of effort and the no shock bonus are also illustrated and summarized in Figure 3. The figure charts $e^*(\pi)$ and $b_n^*(\pi)$ for the same, arbitrarily chosen, sequence of shock and no shock periods as Figure 2. Once again, red squares indicate periods in which the firm has been hit by a shock and blue dots indicate periods in which the firm has not been hit by a shock. The figure shows that $b_n^*(\pi)$ follows the same cycles as expected profits π and that the evolution of $e^*(\pi)$ mirrors that of $b_n^*(\pi)$.

We can now establish our first proposition which characterizes the optimal relational contract.

PROPOSITION 1. *Suppose that Conditions A and B hold. Then the optimal relational contract is characterized by Lemmas 3 − 5 and 7.*

In summary, when Conditions A and B hold, the optimal relational contract has four key characteristics. First, consecutive shock periods lead to a gradual reduction in profits and a gradual increase in the worker's rents. Essentially, the firm can always choose not to pay the worker a bonus, and thus implicitly claim to have been hit by a shock, but if it does so, it also has to promise him higher rents in the future. Second, the firm increases the worker's future rents through both efficient means − by increasing the bonus in the next no shock period − and inefficient means − by allowing the worker to provide less effort until the next no shock period. The failure to pay a bonus therefore limits the firm's ability to operate efficiently in the future. Third, in a period without a shock the firm pays the worker the promised bonus; expected profits then return to their upper bound $\overline{\pi}$ and the worker's rents go to zero. Relationship dynamics are therefore characterized by sluggish downturns but instantaneous recoveries. Finally, the relationship never terminates. Instead, after sufficiently many consecutive shock periods, effort bottoms out at $e^*(\underline{\pi}) > 0$ and the no shock bonus tops out at its maximum level $b_n^*(\underline{\pi})$. Profits are then at their minimum level $\underline{\pi}$ and the worker's rents are at their maximum level $u(\underline{\pi}) - \underline{u}$. Even after arbitrarily many consecutive shock periods, however, it only takes a single no shock period for expected profits to return to $\overline{\pi}$.

The Optimal Relational Contract Does Involve Inefficient Bonus Payments Suppose now that Condition B does not hold so that the optimal relational contract does involve inefficient bonus payments. In the next lemma we characterize $b_s^*(\pi)$.

LEMMA 8. *Suppose that Condition B does not hold. Then there exist three profit levels π_1, π_2, π_3 that satisfy $\underline{\pi} \leq \pi_1 < \pi_2 < \pi_3 < \overline{\pi}$ such that (i.) the optimal bonus $b_s^*(\pi)$ that the firm pays in a shock period is strictly positive if and only if $\pi \in (\pi_1, \pi_3)$. And, (ii.), when $b_s^*(\pi)$ is strictly positive it is non-monotonic in π; in particular*

$$\frac{\mathrm{d}b_s^*(\pi)}{\mathrm{d}\pi} > 0 \quad \textit{for all } \pi \in [\pi_1, \pi_2] \quad \textit{and} \quad \frac{\mathrm{d}b_s^*(\pi)}{\mathrm{d}\pi} < 0 \quad \textit{for all } \pi \in [\pi_2, \pi_3].$$

The lemma is illustrated in Figure 4 which charts $b_s^*(\pi)$ for the same, arbitrarily chosen sequence of shock and no shock periods as Figure 2 and 3. Once again, red squares indicate shock periods and blue dots indicate no shock periods. The figure shows that if expected profits are sufficiently large, the firm does not pay a bonus in a shock period. If, however, the firm is hit by sufficiently many consecutive shocks − and expected profits are therefore sufficiently small − the firm does have to pay a bonus even if it is hit by a shock. Initially, additional shocks then lead to a gradual increase in the shock state bonus. Eventually, however, eventually, however, further shocks actually reduce the shock state bonus until it reaches $b_s^*(\underline{\pi}) \geq 0$, where it stays until the firm reaches a period in which it is not hit by a shock.

To understand the evolution of $b_s^*(\pi)$, recall that the production function is characterized by decreasing returns to scale. As the firm is hit by consecutive shocks, therefore, effort reductions become an increasingly inefficient way of increasing the worker's rents. If the magnitude of the shock α is sufficiently small − so that Condition B does not hold − the firm then limits further effort reduction by promising to pay the worker even if it is hit by a shock. Initially, the shock state bonus is sufficiently small that the reneging constraint $\mathrm{RC_S}$ is not binding. If the firm is hit by sufficiently many consecutive shocks, however, the shock state shock state bonus becomes sufficiently large for the $\mathrm{RC_S}$ constraint to become binding. As a result, the shock state bonus actually decreases, as the firm is hit by additional shocks.

Having determined the optimal bonus in a shock period, we can once again complete the characterization of the optimal relational contract by working out the optimal bonus in a shock state. To so, we simply substitute $\pi_n^*(\pi)$, $\pi_s^*(\pi)$, and $b_s^*(\pi)$ into the $\mathrm{IC_N}$ constraint to obtain the following lemma.

LEMMA 9. *Suppose that Condition B does not hold. Then the optimal bonus in a no shock period is given by*

$$b_n^*(\pi) = b_s^*(\pi) + \frac{\delta}{1-\delta}\left(\overline{\pi} - \pi_s^*(\pi)\right).$$

The next proposition summarizes the characterization of the optimal relational contract when Condition B does not hold.

PROPOSITION 2. *Suppose that Condition A holds but that Condition B does not. Then the optimal relational contract is characterized by Lemmas 3 - 5, 8, and 9.*

In summary, whether or not Condition B is satisfied, the relationship dynamics are essentially the same. The only difference is that if Condition B is not satisfied, and expected profits are sufficiently small, the firm has to pay a bonus even if it has been hit by a shock.

5.2 The Optimal Relational Contract When Condition A Does Not Hold

So far we have focused on situations in which Condition A is satisfied, that is, we focused on situations in which

$$\alpha \geq \frac{\theta}{1-\theta}, \tag{A}$$

where α is the magnitude of the shock and θ its frequency. We now discuss the optimal relational contract when Condition A is not satisfied and argue that it is very similar to the one described above. We discuss the optimal relational contract informally and relegate the formal analysis to the Appendix.

Recall that when Condition A holds, the payoff frontier is everywhere downward sloping. This does not need to be the case when Condition A does not hold. In particular, the payoff frontier may then be maximized at an interior $\pi_m \in (\underline{\pi}, \overline{\pi})$ and thus be upward sloping for $\pi < \pi_m$. Since the payoff frontier is upward sloping for $\pi < \pi_m$, equilibria in which first period profits are strictly less than π_m are Pareto dominated by the equilibrium in which in the first period profits are equal to π_m and the worker's payoff is equal to $u(\pi_m)$. We therefore focus on equilibria in which first period payoffs are given by π and $u(\pi)$, where $\pi \geq \pi_m$. We will see below the for any such equilibrium, profits remain larger than π_m at the beginning of every subsequent period. Also, for any such equilibrium, payoffs can be sustained by pure strategies.

Suppose then that the firm and the worker coordinate on an equilibrium in which first period payoffs are given by π and $u(\pi)$, where $\pi \geq \pi_m$ and $\pi_m \in [\underline{\pi}, \overline{\pi})$. In contrast to the previous section, the optimal relational contract now depends on the level of first period profits. In particular, the optimal relational contract depends on whether first period profits are above a critical value $\pi_a \in [\pi_m, \overline{\pi})$, where, for reasons that will become apparent shortly, the subscript "a" stands for "absorbing."

If the firm and the worker do coordinate on an equilibrium in which first period profits are weakly larger than π_a, the optimal relational contract is very similar to the one discussed in the previous section. Indeed, the only substantive difference then arises when the firm has been hit by a large number of consecutive shocks. Recall that if Condition A is satisfied and the firm is hit by a large number of consecutive shocks, profits bottom out at the lowest feasible level $\underline{\pi}$. If Condition A is not satisfied, profits instead bottom out at $\pi_a \geq \underline{\pi}$. In any equilibrium in which first period profits are weakly larger than π_a, profits therefore remain weakly larger than π_a at the beginning of every subsequent period. With this exception, the optimal relational is qualitatively identical to the one described in the previous section.

If the firm and the worker instead coordinate on an equilibrium in which first period profits are strictly smaller than π_a (but, for the reasons discussed above, still larger than π_m), the optimal relational contract is somewhat different from the one described in the previous section. Profits then always increase initially, whether or not the firm has been hit by a shock. And they continue to increase until profits are larger than π_a. Once profits are larger than π_a, however, the optimal relational contract plays out in exactly the same way as described in the previous paragraph. Once profits are larger than π_a, therefore, profits never fall below π_a again. Except for the initial increase in profits, and the fact that profits never fall below π_a, the optimal relational contract is identical to the one described in the previous section.

6 Discussion

In this section we revisit a few of the key features and assumptions our model and examine them in more detail.

6.1 The Inability to Achieve First Best

A notable feature of the optimal relational contract is that it is always inefficient. In other words, first best can never be sustained, even when the firm and the worker are very patient and when shocks are rare and small. At first this may be somewhat surprising since only the firm has private information − and may thus need to be punished on the equilibrium path − and since the firm can be punished efficiently by having to make payments in no shock periods in which its opportunity costs are low. The reason why the optimal relational contract is always inefficient is that the worker can never be sure that the firm's opportunity costs are low. The firm can therefore always avoid an efficient punishment that only calls for higher payments in a future no shock period by falsely claiming that it has not been hit by a shock.

To see this, suppose that whenever the firm is not hit by a shock, there is some probability $p \in (0, 1)$ with which it becomes publicly known that the firm's opportunity costs are indeed low.[5] In the Appendix, we show that first best can then be achieved for sufficiently high discount rates. In this setting, whenever the firm is hit by a shock, it does not pay the worker a bonus but promises him a bigger bonus in the next period in which it is publicly known that its opportunity costs are low. There is then no need to punish the firm inefficiently, by reducing effort or increasing bonus payments in shock periods.

While public information about low opportunity costs may allow the firm and the worker to sustain first best, public information about high opportunity costs do not. To see this, suppose that whenever the firm is hit by a shock, there is some probability $q \in (0, 1)$ with which it becomes publicly known that the firm's opportunity costs are indeed high. In the Appendix we show that first best can then still not be achieved. Essentially, while public information about low opportunity costs allows the firm to be punished efficiently, public information about high opportunity costs does not. An important implicit assumption in our setting is therefore that the worker can never prove that the firm's opportunity costs are low. In contrast, the assumption that the firm can never prove that its opportunity costs are high is less important.

6.2 The Public Information Benchmark

Our model has two key ingredients: asymmetric information and the lack of formal contracts. Without either ingredient, the relationship between the firm and the worker would develop very differently from the way described above. To see this, we now compare the optimal relational contract in our setting to what it would be if shocks were publicly observed. And in the next section we then compare the optimal relational contract to the contract that the firm and the worker would agree to if formal contracting were feasible.

Suppose that shocks are publicly observed. And to facilitate the exposition, suppose that the

[5] If $p = 0$, the model is the same as the one that we examined in the previous section. And if $p = 1$ the occurance of the shock is publicly observed and there is no need for the firm to be punished on the equilbrium path. We discuss this public information benchmark in Section Z.

firm simply cannot pay the worker in a shock period, that is, suppose that $\alpha \rightarrow \infty$. Proposition Z in the Appendix then characterizes the optimal relational contracts for any first period payoffs that are on the payoff frontier. For any first period payoffs that are on the payoff frontier there may now be multiple relational contracts that are optimal. Whenever there are multiple relational contracts that are optimal, however, they induce the same effort level and thus generate the same joint surplus. We therefore do not lose any insight by focusing on one specific relational contract that is optimal, as we do next. In that contract, the firm's profits always weakly increase over time. In particular, for any profit level $\pi \in [\underline{\pi}, \overline{\pi}]$, the continuation profits are given by $\pi_n = \overline{\pi}$ and $\pi_s(\pi) = \pi$. Moreover, effort $e(\pi)$ is strictly increasing in π unless it is already at its first best level, in which case further increases in π have no effect on effort. Profits, effort, and joint surplus therefore increase over time until they reach their highest feasible levels. If the firm and the worker are patient enough, those highest feasible levels are equal to first best and otherwise they are strictly below first best. In either case, once profits, effort, and joint surplus are at their highest feasible levels, they stay there forever.

Relationship dynamics are therefore very different when shocks are publicly rather than privately observed. Essentially, when shocks are publicly observed, time is your friend: profits, effort, and joint surplus increase over time until they reach their largest levels and then stay there indefinitely. In contrast, when shocks are privately observed, time may be your foe: whenever profits, effort, and joint surplus have reached their highest levels, they are certain to decrease again, at least temporarily.

We can also interpret our model as a trust game in which the worker's effort is a measure of the degree to which he trusts the firm. The model with publicly observed shocks then captures the intuition that relationships get better over time because it takes time to build trust. In contrast, when shocks are privately observed, relationships may get worse over time because the burden of past promises erodes trust. In principle, these different predictions about how relationships evolve are testable in an experimental setting.

6.3 The Full Contracting Benchmark

Suppose next that the firm can commit to a formal contract. In particular, suppose that in any period t the firm first observes the state $\Theta_t \in \{n, s\}$ − that is, it observes whether it has been hit by a shock or not − and then makes an announcement $m_t \in \{n, s\}$ about the state. Suppose also that before the first period, the firm can commit to a contract that, for any period t, maps its announcements $(m_1, m_2, ..., m_t)$ into the bonus b_t that the firm has to pay the worker at the end of period t. In this section we show that the optimal contract in this setting induces very different behavior than the optimal relational contract in our setting.

Notice first that even if the firm can commit to a formal contract, the firm and the worker cannot achieve first best. To achieve first best, the contract would have to induce the worker to always provide first best effort. At the same time, the contract require the firm to only pay the worker when it admits to not having been hit by a shock. Naturally, the firm would then never admit to having been hit by a shock.

While formal contracting does not allow the firm and the worker to achieve first best, it does allow them to come arbitrarily close to achieving first best. To see this, let $\tau(t)$ denote the number of consecutive periods immediately preceding t in which the firm did not pay the worker. So, for instance, if we were currently in period ten and the firm did not pay the worker in period nine but did pay him in period eight, then $\tau(10) = 1$. Now consider a contract with three features. First, the contract asks the worker to always provide first best effort. If the worker ever does not provide first best effort, the firm will never again pay him. Second, the contract specifies that if, in period t, the firm announces that it has not been hit by a shock, it pays the worker a bonus

$$b_t = \left(1 + \frac{1}{\delta} + \frac{1}{\delta^2} + ... + \frac{1}{\delta^{\tau(t)}}\right) \left(\underline{u} + c\left(e_{fb}\right)\right).$$

Third, the contract specifies a number $T \geq 1$ that determines how much the firm has to pay the worker whenever it announces that it has been hit by a shock. In particular, if, in period t, the firm announces that it has been hit by a shock and if $\tau(t) < T$, then the firm does not have to pay the worker. If, however, $\tau(t) = T$, then the firm has to pay the worker

$$b_t = \left(1 + \frac{1}{\delta} + \frac{1}{\delta^2} + ... + \frac{1}{\delta^{T}}\right) \left(\underline{u} + c\left(e_{fb}\right)\right).$$

Proposition LT1 in the Appendix shows that under such a contract, the worker always provides first best effort and the firm always announces the state truthfully. Essentially, under this contract the firm will have to pay the worker $\left(\underline{u} + c\left(e_{fb}\right)\right)$ per period, independent of its announcements. In other words, while the firm's announcements may affect the timing of payments, they do not affect their net present value. Faced with this contract, it is therefore an optimal response for the firm to truthfully reveal the state.

The problem with this contract, of course, is that it induces inefficient bonus payments whenever the firm is hit by shocks for T consecutive periods. Notice, however, that the firm and the worker can reduce this inefficiency by agreeing to a larger T. In fact, the firm and the worker can come arbitrarily close to first best by agreeing to an arbitrarily large T. The optimal formal contract therefore induces very different behavior than the optimal relational contract. In summary, both asymmetric information and the lack of formal contracts are therefore crucial for the relationship dynamics that are predicted by our model.

7 What Happens if the Firm is Liquidity Constrained?

So far we have assumed that the firm can always pay the worker any bonus it wants to, even if its opportunity costs of doing so may be high. In other words, we have assumed that the firm is not liquidity constrained. We now relax this assumption. Doing so sheds more light on what drives some of the key features of optimal relational contract that we described above. It also modifies the optimal relational contract in ways that are consistent with some of the anecdotal evidence in the Introduction.

Specifically, we now make two changes to our model. First, we now assume that the firm simply cannot pay the worker in a shock period, that is, we assume that α is equal to infinity. We make this assumption for simplicity. Second, we now assume that if the firm realizes output $y(e)$ and is not hit by a shock, it can pay the worker at most $(1+m)\,y(e)$, where the parameter $m \geq 0$ captures the extent to which the firm is liquidity constrained. Moreover, we assume that the liquidity constraint is binding when expected profits π are at their lower bound $\underline{\pi}$. If this were not the case, the liquidity constraint would never be binding and the optimal relational contract would be the same as the one described above.

In what follows we summarize the key features of the optimal relational contract that we derive formally in the Appendix. Recall that π_0 denotes the largest profit π such that for all $\pi \geq \pi_0$, the payoffs on the payoff frontier can be sustained by pure strategies. We saw above that when the firm is not liquidity constrained and α is sufficiently large, then $\pi_0 = \underline{\pi}$. When the firm is liquidity constrained, in contrast, it can be the case that $\pi_0 > \underline{\pi}$ even when $\alpha \to \infty$. We return to this case at the end of the section but for now we focus on the optimal relational contract when $\pi_0 = \underline{\pi}$. In that case, the optimal relational contract is fully described by the optimal continuation profits $\pi_s^*(\pi)$ and $\pi_n^*(\pi)$, the optimal no shock bonus $b_n^*(\pi)$, and optimal effort $e^*(\pi)$.

Consider first a period in which the firm expects to make profits $\pi \in [\underline{\pi}, \overline{\pi}]$ and is then hit by a shock. Tomorrow's expected profits will then be strictly smaller than today's, unless today's expected profits are already at their lower bound $\underline{\pi}$, in which case they stay there. In other words, $\pi_s^*(\pi) < \pi$ if $\pi > \underline{\pi}$ and $\pi_s^*(\underline{\pi}) = \underline{\pi}$. As in the model without liquidity constraints, therefore, consecutive shock periods lead to a gradual reduction in expected profits until they reach their lower bound $\underline{\pi}$. And once expected profits have reached their lower bound $\underline{\pi}$, they stay there until the next no shock period.

Next, consider a period in which the firm expects to make profits $\pi \in [\underline{\pi}, \overline{\pi}]$ and is not hit by a shock. Tomorrow's expected profits will then be strictly larger than today's, unless today's expected profits are already at their upper bound $\overline{\pi}$, in which case they stay there. In other words,

$\pi_n^*(\pi) > \pi$ if $\pi < \overline{\pi}$ and $\pi_n^*(\overline{\pi}) = \overline{\pi}$. As in the model without liquidity constraints, therefore, consecutive no shock periods increase expected profits until they reach their upper bound $\overline{\pi}$ after which they stay there. In contrast to the model without liquidity constraints, however, it may now take more than one no shock period for expected profits to reach their upper bound $\overline{\pi}$. In particular, we now have that $\pi_n^*(\pi) < \overline{\pi}$ if $\pi < \pi_1$ and $\pi_n^*(\pi) = \overline{\pi}$ otherwise, where $\pi_1 \in (\underline{\pi}, \overline{\pi})$ is defined in the Appendix. If profits are sufficiently small, therefore, it takes at least two consecutive no shock periods for expected profits to reach $\overline{\pi}$. Essentially, for the worker to agree to move to the equilibrium in which expected profits are maximized, the firm has to compensate him for the corresponding loss in the his rents. If the firm is not liquidity constrained, it can compensate the worker with a single, large bonus payment. But if the firm is liquidity constrained, it may have to spread the bonus payment over multiple periods. In summary, while recoveries are instantaneous in the absence of liquidity constraints, they can be sluggish when the firm is liquidity constrained.

Together the two continuation profits that we characterized in the previous two paragraphs determine the optimal no shock bonus. In particular, substituting $\pi_s^*(\pi)$ and $\pi_n^*(\pi)$ into the IC$_N$ constraint and making use of the fact that $b_s^*(\pi) = 0$, we get that $(1-\delta) b_n^*(\pi) = \delta(\pi_n^*(\pi) - \pi_s^*(\pi))$. Whenever the liquidity constraint is binding, therefore, a reduction in expected profits may lead to a reduction in the optimal no shock bonus.

Finally, consider $e^*(\pi)$, the worker's optimal effort in a period in which the firm expects to make profits $\pi \in [\underline{\pi}, \overline{\pi}]$. In the absence of liquidity constraints, a reduction in expected profits always leads to a reduction in effort. In contrast, when the firm is liquidity constrained, a reduction in expected profits can actually lead to an increase in effort. The reason is that an increase in effort now has the additional benefit of relaxing the liquidity constraint. A reduction in expected profits may therefore require an increase in effort to ensure that the firm can pay the worker a larger bonus in the next no shock period. This is in line with the example of Lincoln Electric in the Introduction. There management responded to a reduction in profits, and the resulting inability to pay the workers their bonus, by asking those workers to work even harder. And the workers agreed to work even harder because they understood that this would allow Lincoln Electric to pay them their next bonus. Specifically, we now have that $e^*(\pi)$ is strictly decreasing in π if $\pi \in [\pi_1, \pi_2]$, where $\pi_2 \in [\pi_1, \overline{\pi}]$ is defined in the Appendix. There we also provide a sufficient condition for this region to exist, that is, for $\pi_2 > \pi_1$. When this region does exist and profits are within this region, the firm is liquidity constrained but $\pi_n^*(\pi)$ is still at its upper bound $\overline{\pi}$. If, instead, $\pi > \pi_2$, the liquidity constraint is not binding and $e^*(\pi)$ in increasing in π, just as in the main model. And, finally, if $\pi < \pi_1$ the effect of an increase in π on $e^*(\pi)$ is ambiguous. In summary, while a reduction in expected profits always leads to a reduction in effort in the absence of liquidity constraints, a reduction in expected profits can lead to an increase in effort when the firm is liquidity constrained.

So far we have focused on the case in which termination is not part of the optimal relational contract, that is, we have focused on the case in which $\pi_0 = \underline{\pi}$. To conclude this section, suppose instead that $\pi_0 > \underline{\pi}$. For any $\pi \in [\pi_0, \overline{\pi}]$ the optimal relational contract is then essentially same as the one described in the last four paragraphs. The only important difference is that after sufficiently many consecutive shock periods expected profits will now be strictly below π_0. And once expected profits are strictly below π_0, the firm and the worker randomize between separating and playing the pure strategies that deliver the payoffs π_0 and $u(\pi_0)$. At this point, even if the firm and the worker are lucky and do not have to separate immediately, they are certain to separate if they are hit by another shock in the next period. This is so since $\pi_s^*(\pi_0) = \underline{\pi}$. The relationship is therefore certain to terminate after a finite number of consecutive shocks. This, of course, is in contrast to optimal relational contract without liquidity constraints which never calls for termination when α is sufficiently large and, in particular, when $\alpha \to \infty$. To understand this difference, recall that when the firm is liquidity constrained and hit by a number of consecutive shocks, the continuation profits in the next no shock period $\pi_n^*(\pi)$ are very low. The firm's reward for admitting to not having been hit by a shock is therefore very limited. To induce the firm to still be truthful, the worker therefore needs to increase the punishment for claiming to have been hit by a shock. And when profits are already very small, the only way to do so is to increase the threat of termination.

In summary, allowing for the firm to be liquidity constrained changes the optimal relational contract in three main ways. First, recoveries can now be sluggish rather than instantaneous. Second, the failure to pay the worker a bonus can now lead to more effort rather than less. And third, termination can now be part of the optimal relational contract even when α is very large.

8 Conclusion

Sustaining successful relationships is difficult in a changing and opaque world. Actions that create value can nevertheless destroy trust. This paper studies how to best structure relational contracting in the presence of incomplete information and explores its implications on the dynamics of the relationship. We develop a relational contracting model between a worker and a firm who is privately informed about shocks to its cost of transfer. We show that the efficiency of the relationship is bounded away from the first-best level. More efficient actions in the current time – lower bonus payment when the transfer is costly – come at the cost of inefficient actions in the future - lower effort from the worker for example. The value of the relationship cycles over time, exhibiting sluggish downturns and immediate recovery. When the firm is also liquidity constrained, the efficiency of the relationship can take longer to recover. More interestingly, the worker may increase his effort after a shock state, gambling for more bonuses in the near future. But such gambles can lead to the demise of the relationship in the long run.

We have cast the model in the context of a firm-worker relationship. The main ingredients of the model—repeated interaction, limited commitment, and hidden information—are also relevant to many other important economic environments. One example is the lending relationship between an entrepreneur and a bank. The entrepreneur can have hidden information of his marginal value of money, and the bank can adjust its future terms of lending based on the past payment history of the entrepreneur. Another example is the informal insurance relationships among farmers in developing countries. There's some evidence that the farmer's income is hidden information, see for example, Kinnan (2010). While most of the literature has focused on moral hazard or insurance issues separately in this context, our model suggests that these issues are related since insurance decisions affect future production choices. Further research in this area is needed.

9 Appendix

9.1 PPE Payoffs

This part of the appendix provides the necessary technical background for deriving the main results. The key result in this subsection is that the PPE payoff is differentiable for all $\pi \in [\underline{\pi}, \overline{\pi}]$.

LEMMA A1: Let $\overline{\pi}$ be the maximum PPE payoff of the principal. The PPE payoff set E is given by

$$E = \{(\pi', u') : \pi' \in [\underline{\pi}, \overline{\pi}], u' \in [\underline{u}, u(\pi')]\}.$$

In addition, u is concave, and

$$u(\overline{\pi}) = \underline{u}.$$

Proof. First, note that the payoff pair $(\underline{\pi}, \underline{u})$ (meaning that the principal's normalized expected payoff is $\underline{\pi}$ and the agent's normalized expected payoff is $\underline{u}$) is in the PPE payoff set. This payoff is supported by the equilibrium in which on the equilibrium path, the principal and the agent does not start a relationship, and off the equilibrium path, the agent puts in effort $e = 0$, the principal always pay $b_s = b_n = 0$, and both the principal and the agent does not the start the relationship in the future.

Second, to see that $u(\overline{\pi}) = \underline{u}$, suppose to the contrary that $u(\overline{\pi}) > \underline{u}$. Since $(\overline{\pi}, u(\overline{\pi}))$ is an extremal point of the PPE, it is sustained by pure strategy in period 1. Let $e(\overline{\pi})$ be the agent's effort associated with $\overline{\pi}$ in period 1. Now modifying this equilibrium strategy by increasing $e(\overline{\pi})$ to $e(\overline{\pi}) + \varepsilon$ for small enough ε and keep everything else the same. This change results a strategy that is also a PPE. But the new PPE gives the principal a higher payoff than $\overline{\pi}$. This contradicts the definition of $\overline{\pi}$.

Now the availability of the public randomization device implies that any payoff on the line segment between $(\underline{\pi}, \underline{u})$ and $(\overline{\pi}, \underline{u})$ can be supported as a PPE payoff. It then follows that any payoff (π, u') can be obtained from the randomization between $(\pi, \underline{u})$ and $(\pi, u(\pi))$ for all $u' \in [\underline{u}, u(\pi)]$. The concavity of u follows directly from the availability of the public randomization device. Finally, it is clear that any PPE payoff pair must give the principal at least $\underline{\pi}$ and the agent at least $\underline{u}$. This finishes the proof. ■

LEMMA A2: There exists a π_0 such that for all $\pi \in [\pi_0, \overline{\pi}]$, $u(\pi)$ can be sustained by pure strategy.

Proof. *It suffices to show that for any $\pi_1 < \pi_2$, if both $u(\pi_1)$ and $u(\pi_2)$ are sustained by pure strategies (other than taking the outside option), then for every $\pi \in (\pi_1, \pi_2)$, $u(\pi)$ can be sustained by a pure strategy.* In other words, if both $u(\pi_1)$ and $u(\pi_2)$ are sustained by pure strategies (other than the outside option), we don't need randomization in between.

Let $e_i, b_{s_i}, b_{n_i}, \pi_{s_i}, \pi_{n_i}$, $i = 1, 2$ be the associated effort and continuation payoffs. Suppose

$$\pi = \alpha\pi_1 + (1 - \alpha)\pi_2, \quad \text{for some } \alpha \in (0, 1).$$

Let e be the effort level such that

$$y(e) = \alpha y(e_1) + (1 - \alpha)y(e_2).$$

Given that y is increasing and concave, we know that

$$y(\alpha e_1 + (1 - \alpha)e_2) \geq \alpha y(e_1) + (1 - \alpha)y(e_2),$$

so

$$e \leq \alpha e_1 + (1 - \alpha)e_2$$

Also, let

$$\begin{aligned}
b_s &= \alpha b_{s_1} + (1-\alpha) b_{s_2} \\
b_n &= \alpha b_{n_1} + (1-\alpha) b_{n_2} \\
\pi_s &= \alpha \pi_{s_1} + (1-\alpha) \pi_{s_2}; \\
\pi_n &= \alpha \pi_{n_1} + (1-\alpha) \pi_{n_2}.
\end{aligned}$$

One can check that this set of e, b_s, b_n, π_s, and π_n satisfy all of the constraints.

In addition,

$$\begin{aligned}
& u(\pi) - (\alpha u(\pi_1) + (1-\alpha) u(\pi_2)) \\
= \ & \alpha c(e_1) + (1-\alpha) c(e_2) - c(e) \\
\geq \ & 0,
\end{aligned}$$

since c is increasing and convex and $e \leq \alpha e_1 + (1-\alpha) e_2$. ■

LEMMA A3: Let $(\pi, u(\pi))$ be a PPE payoff sustained by pure strategy in period 1. Let (π_s, u_s) and (π_n, u_n) be the continuation payoffs following the shock and nonshock state respectively. Then

$$\begin{aligned}
u_s &= u(\pi_s); \\
u_n &= u(\pi_n).
\end{aligned}$$

Proof. Suppose to the contrary that $u_s < u(\pi_s)$. Consider a new payoff pair that specify the same actions and continuation payoffs as $(\pi, u(\pi))$ except that the continuation payoff (π_s, u_s) is changed to $(\pi_s, u_s + \varepsilon)$ for some small positive ε. This change does not violate any of the constraints, and therefore the new payoff pair from this decomposition is also a PPE payoff. This new payoff pair again gives the agent a payoff of $u(\pi) + \delta(1-\theta)\varepsilon$ and this violates the definition of $u(\pi)$.

Identical proof can be used to show that $u_n = u(\pi_n)$. ■

LEMMA A4:

$$\delta (\pi_n - \pi_s) = (1-\delta)(b_n - b_s) \tag{IC\_N}$$

Proof. Recall that $b_n \geq b_s$ and in addition, $\mathrm{IC_N}$ implies $\delta(\pi_n - \pi_s) \geq (1-\delta)(b_n - b_s)$. Suppose instead we have $\delta(\pi_n - \pi_s) > (1-\delta)(b_n - b_s)$. Let

$$\begin{aligned}
\pi_s' &= \pi_s + \theta \varepsilon; \\
\pi_n' &= \pi_n - (1-\theta)\varepsilon.
\end{aligned}$$

for some small ε (while keeping b_s, b_n and e). The budget constraint and the truthtelling conditions remain satisfied under the new set of continuation payoffs (π_s' and π_n').

In addition, the new set of continuation payoffs satisfy

$$\theta\pi_s' + (1-\theta)\pi_n' = \theta\pi_s + (1-\theta)\pi_n$$

This implies that the agent's value under the new continuation payoffs is

$$\begin{aligned} & (1-\delta)\left(\theta b_s + (1-\theta) b_n - c(e)\right) + \delta(\theta u\left(\pi_s'\right) + (1-\theta) u\left(\pi_n'\right)) \\ \geq \ & (1-\delta)\left(\theta b_s + (1-\theta) b_n - c(e)\right) + \delta(\theta u(\pi_s) + (1-\theta) u(\pi_n)), \end{aligned}$$

where the inequality follows from the concavity of u. ■

9.1.1 Differentiability of the Payoff Frontier

Now we proceed to show that u is differentiable in all of its domain. Define π_m as the largest point at which $u(\pi)$ is maximized. Note that $\pi_m \geq \pi_0$. We first show that for $\pi > \pi_m$, $e > 0$ and u is differentiable.

LEMMA A5: For $\pi > \pi_m$, $e(\pi) > 0$, u is differentiable and

$$u'(\pi) = -\frac{c'}{y'}.$$

Proof. Consider $\pi + \varepsilon$ for some $\varepsilon > 0$. Then choose $e\prime > e(\pi)$ such that

$$(1-\delta)\, y(e\prime) - (\pi + \varepsilon) = (1-\delta)\, y(e) - \pi.$$

Let $\pi_s' = \pi_s$ and $\pi_n' = \pi_n$ be unchanged. Then this set of e', π_s', and π_n' is feasible for π (in the sense that the constraints are satisfied.)

By the definition of the payoff frontier, we have

$$u(\pi + \varepsilon) \geq u(\pi) - (1-\delta)\left(c(e\prime) - c(e)\right)$$

This implies that

$$u_+'(\pi) \geq -\frac{c'}{y'}.$$

Now for $\pi > \pi_m$, we have

$$u_+'(\pi) < 0,$$

and this implies that $\frac{c'}{y'} > 0$, or $e(\pi) > 0$.

Given $e(\pi) > 0$, we can repeat the argument above for $\varepsilon < 0$, we have

$$u'_-(\pi) \leq -\frac{c'}{y'}.$$

Since $u'_-(\pi) \geq u'_+(\pi)$, we then have

$$u'_+(\pi) = u'_-(\pi) = -\frac{c'}{y'}$$

■

Next, we show that RCs constraint is binding when $u'_+(\pi) > -\frac{1}{1+\alpha\theta}$.

LEMMA A6: If $u'_+(\pi) > -\frac{1}{1+\alpha\theta}$, *then* $RC - S$ *is binding.*

Proof. As we move from π to $\pi - \varepsilon$, we let $e\prime = e(\pi)$, $\pi_s\prime = \pi_s$, and $\pi'_n = \pi_n$. This set of effort and continuation payoffs relax the non-negativity constraint, it hardens the RCs constraint (but this does not matter since it isn't binding in this first place.) Finally, this set of values increase u by $\frac{1}{1+\alpha\theta}$. In other words,

$$u(\pi - \varepsilon) - u(\pi) > \frac{\varepsilon}{1 + \alpha\theta}.$$

As ε goes to 0, we have the desired result. ■

The next lemma shows that when $\theta \leq \frac{\alpha}{1+\alpha}$, the payoff frontier peaks at $\pi_m = \underline{\pi}$. Since u is differentiable for all $\pi > \pi_m$, this establishes the differentiability of u when $\theta \leq \frac{\alpha}{1+\alpha}$.

LEMMA A7: If $\theta \leq \frac{\alpha}{1+\alpha}$,

$$\pi_m = \underline{\pi}.$$

Proof. Suppose to the contrary that $\pi_m > \underline{\pi}$. By the previous two lemmas, we have $e(\pi_m) = 0$ and RCs binds. First, we have from IC-N that

$$\pi_s = \frac{1}{\delta}(\pi_m + (1 - \delta)(1 + \theta\alpha) b_s) > \pi_m.$$

In addition, when RCs binds and when $\pi_s > \pi_m$, we must then have $b_s > 0$. This is because if RCs binds and $b_s = 0$, we would get $\pi_s = \underline{\pi}$, contradicting the above.

Moreover, given that RCs binds, we have (using IC-N) that

$$\delta\pi_s = \frac{1 + \alpha}{\alpha(1 - \theta)}\pi_m - \frac{\delta(1 + \alpha\theta)}{\alpha(1 - \theta)}\underline{\pi}.$$

Now suppose the principal's payoff decreases from π^* to $\pi^* - \varepsilon$, one can keep the effort $e = 0$, decreases continuation payoff of the shock state from π_s to $\pi_s - \frac{1+\alpha}{\delta\alpha(1-\theta)}\varepsilon$, keep π_n, and adjust b_s accordingly to keep the IC-N. This set of actions and continuation payoffs are feasible for small ε because we maintain RCs and IC-N by choice and the only constraint it hurts is $b_s \geq 0$, which is slack in the first place by above.

In addition, the agent's expected payoff from the change above cannot exceed $u(\pi_m - \varepsilon)$ by the definition of u as the payoff frontier. Sending ε to 0, we get

$$u'_-(\pi_m) \leq \frac{\theta - \alpha(1-\theta)}{\alpha(1-\theta)} + \frac{\theta(1+\alpha)}{\alpha(1-\theta)} u'_-(\pi_s).$$

But this is a contradiction because $u'_-(\pi_m) \geq 0$ by the definition of π_m. However, $u'_-(\pi_s) < 0$ (because $\pi_s > \pi_m$) and $\frac{\theta-\alpha(1-\theta)}{\alpha(1-\theta)} < 0$ (by assumption), so the right hand side is negative. This is a contradiction. ■

LEMMA A8: If $\theta > \frac{\alpha}{1+\alpha}$, and $\pi_m > \underline{\pi}$, then for π with $u'_+(\pi) > \frac{\alpha-(1+\alpha)\theta}{(1+\alpha)\theta-\alpha(1-\theta)}$,

$$\pi_s > \pi.$$

Proof. Consider a payoff π such that $u'_+(\pi) > -\frac{1}{1+\alpha\theta}$. Then $RC - S$ is binding in this case. Now suppose the principal's payoff increases from π to $\pi + \varepsilon$. One can keep the effort e, increases continuation payoff of the shock state from π_s to $\pi_s + \frac{1+\alpha}{\delta\alpha(1-\theta)}\varepsilon$, keep π_n, and adjust b_s accordingly to keep the IC-N. This set of actions and continuation payoffs are feasible for small ε because we maintain RCs and IC-N by choice and it relaxes the non-negativity constraint $b_s \geq 0$.

In addition, the agent's expected payoff from the change above cannot exceed $u(\pi_m + \varepsilon)$ by the definition of u as the payoff frontier. Sending ε to 0, we get

$$u'_+(\pi) \geq \frac{\theta - \alpha(1-\theta)}{\alpha(1-\theta)} + \frac{\theta(1+\alpha)}{\alpha(1-\theta)} u'_+(\pi_s).$$

Given $\theta > \frac{\alpha}{1+\alpha}$, the above implies that when $u'_+(\pi) > \frac{\alpha-(1+\alpha)\theta}{(1+\alpha)\theta-\alpha(1-\theta)}$ we have $u'_+(\pi_s) < u'_+(\pi)$, and, thus.

$$\pi_s > \pi.$$

■

Combining the two lemmas above, we now have the following proposition.

Proposition A1: u is differentiable for all $\pi \in [\underline{\pi}, \overline{\pi}]$.

Proof. When $\pi_m = \underline{\pi}$, we have $e(\pi) > 0$ for all $\pi > \underline{\pi}$, and, thus, we have $u'(\pi) = -\frac{c'}{y'}$ for all π. By the two proceeding lemmas, the only remaining case is for $\theta > \frac{\alpha}{1+\alpha}$ and that $\pi_m > \underline{\pi}$.

Now suppose to the contrary that there exists a payoff level π at which $u'_+(\pi) < u'_-(\pi)$. Note that in this case we must have $u'_+(\pi) \geq 0$ (because otherwise u is differentiable with $u' = -\frac{c'}{y'}$). Given $\theta > \frac{\alpha}{1+\alpha}$, we then have $u'_+(\pi) > \frac{\alpha-(1+\alpha)\theta}{(1+\alpha)\theta-\alpha(1-\theta)}$, and by the lemma above, we have $\pi_s > \pi$ and that

$$u'_+(\pi) \geq \frac{\theta - \alpha(1-\theta)}{\alpha(1-\theta)} + \frac{\theta(1+\alpha)}{\alpha(1-\theta)} u'_+(\pi_s).$$

Note also that we have $b_s > 0$ (because the RCs binds at π and that $\pi_s > \underline{\pi}$), then by using the same argument as in Lemma A7, we have

$$u'_-(\pi) \leq \frac{\theta - \alpha(1-\theta)}{\alpha(1-\theta)} + \frac{\theta(1+\alpha)}{\alpha(1-\theta)} u'_-(\pi_s).$$

Combining these two inequalities above, we see that when $u'_+(\pi) < u'_-(\pi)$, we then have $\pi_s > \pi$ and that

$$u'_+(\pi_s) < u'_-(\pi_s).$$

Now let π_{nd} be the largest nondiffereniable point. If π_{nd} exists, we then have a contradiction because the argument above implies that $\pi_s(\pi_{nd}) > \pi_{nd}$ is again nondifferentiable. If the largest nondifferentiable point does not exist, let π_{nd} be the supremum of such points. Take a nondifferentiable point π sufficiently close to π_{nd}. The argument above then implies that $\pi_s(\pi)$ is again nondifferentiable and $\pi_s(\pi) \in (\pi, \pi_{nd})$. Since π is sufficiently close to π_{nd}, we have that

$$u'_+(\pi_s) > u'_+(\pi) - \varepsilon$$

for some small ε.

It follows that then that

$$\begin{aligned} u'_+(\pi) &\geq \frac{\theta - \alpha(1-\theta)}{\alpha(1-\theta)} + \frac{\theta(1+\alpha)}{\alpha(1-\theta)} u'_+(\pi_s) \\ &\geq \frac{\theta - \alpha(1-\theta)}{\alpha(1-\theta)} + \frac{\theta(1+\alpha)}{\alpha(1-\theta)} (u'_+(\pi) - \varepsilon). \end{aligned}$$

But for small enough ε, the above implies that $u'_+(\pi) < 0$, which contradicts the nondifferentiability of π. ■

9.2 Main Results

In this subsection, we derive the main results of the paper. To facilitate the proofs, the order that the lemma are proved is not identical to the order that they are stated in the main text but is similar. The main text shows that the total payoff of the relationship satisfies the following constrained optimization problem:

$$\pi + u(\pi) = \max_{e,b_s,\pi_s,\pi_n} (1-\delta)\left(y(e) - c(e)\right) + \theta\delta\left(\pi_s + u(\pi_s)\right) + (1-\theta)\delta\left(\pi_n + u(\pi_n)\right) - (1-\delta)\theta\alpha b_s \tag{2}$$

such that

$$\pi = (1-\delta)\,y(e) + \delta\pi_s - (1-\delta)(1+\theta\alpha)\,b_s. \tag{IC$_{\mathrm{N}}$}$$

$$\delta\pi_s - \delta\underline{\pi} \geq (1-\delta)(1+\alpha)\,b_s. \tag{RC$_{\mathrm{S}}$}$$

$$b_s \geq 0; \tag{Neg$_{\mathrm{S}}$}$$

$$e \geq 0. \tag{Neg$_{\mathrm{e}}$}$$

$$\pi_n \leq \overline{\pi}. \tag{Self-enf}$$

Define the Lagrangian as

$$\begin{aligned} \pi + u(\pi) &= L = (1-\delta)\left(y(e) - c(e)\right) + \theta\delta\left(\pi_s + u(\pi_s)\right) + (1-\theta)\delta\left(\pi_n + u(\pi_n)\right) - (1-\delta)\theta\alpha b_s \\ &\quad + \lambda_1(\pi - (1-\delta)\,y(e) - \delta\pi_s + (1-\delta)(1+\theta\alpha)\,b_s) \\ &\quad + \lambda_2(\delta\pi_s - \delta\underline{\pi} - (1-\delta)(1+\alpha)\,b_s) \\ &\quad + \lambda_3 b_s + \lambda_4 e + \lambda_5(\overline{\pi} - \pi_n). \end{aligned}$$

The key FOCs are those with respect to π_s and b_s :

$$\theta(1 + u'(\pi_s)) - \lambda_1 + \lambda_2 = 0. \tag{FOC-π_s}$$

$$-\theta\alpha + \lambda_1(1+\theta\alpha) - \lambda_2 + \lambda_3 = 0 \tag{FOC-b_s}$$

The envelop condition gives that

$$1 + u'(\pi) = \lambda_1. \tag{envelop}$$

Combining these three conditions give us the following lemma.

LEMMA 1. *The slope of the payoff frontier $u(\pi)$ satisfies*

$$\frac{\mathrm{d}u(\pi)}{\mathrm{d}\pi} > -1 \quad \textit{for all } \pi \in [\underline{\pi}, \overline{\pi}].$$

Proof. By combining the FOC wrt π_s and the envelop condition, we have $\theta(1 + u'(\pi_s)) = 1 + u'(\pi) - \lambda_2 \leq 1 + u'(\pi)$. If $u'(\pi) < -1$, we have $u'(\pi_s) < u'(\pi)$, and this leads to a contradiction at $\pi = \overline{\pi}$. Therefore, we have $u'(\pi) \geq -1$ for all π.

Now suppose $\frac{\mathrm{d}u(\pi)}{\mathrm{d}\pi} = -1$ for some π. Define $\pi' = \min\{\pi : \frac{\mathrm{d}u(\pi)}{\mathrm{d}\pi} = -1\}$. For $\pi \in [\pi', \overline{\pi}]$, we have we have $u'(\pi_s) = -1$ by the previous paragraph, and we also have $u'(\pi_n) = -1$ since $\pi_n \geq \pi_s$. Therefore, once $\pi \in [\pi', \overline{\pi}]$, its continuation payoffs stay in the interval forever.

From Lemma A5, we see that the effort will always be at the first best level. Now if $b_s > 0$ for some $\pi \in [\pi', \overline{\pi}]$, we can lower b_s by ε and lower $y(e)$ by $(1 + \theta\alpha)\varepsilon$ for small enough ε. This change relaxes the $\mathrm{RC_S}$ constraint while keeping all other constraints. For small enough ε, this change increases $\pi + u(\pi)$ by $(1 - \delta)\theta\alpha\varepsilon$ (since $y' = c'$ at first best level of effort), which is a contradiction. This implies that $b_s = 0$ for all $\pi \in [\pi', \overline{\pi}]$.

The above implies that for any $\pi \in [\pi', \overline{\pi}]$, $b_s = 0$, $e = e^{fb}$. Since once $\pi \in [\pi', \overline{\pi}]$, its continuation payoffs stay in the interval forever, the firm always get first-best level of effort, and it has the option of never paying the worker (since $b_s = 0$). This implies that the worker's payoff is less than $-c(e^{fb})$, which is less than his outside option. This is a contradiction. ■

An immediate consequence of Lemma 1 is the following.

LEMMA 3. *The optimal continuation profit in a period without a shock is given by*

$$\pi_n^* = \overline{\pi} \quad \textit{for all } \pi \in [\underline{\pi}, \overline{\pi}].$$

Proof. The FOC wrt to π_n gives that

$$(1 - \theta)\delta\left(1 + u'(\pi_n^*)\right) - \lambda_5 = 0.$$

If $\pi_n^* < \overline{\pi}$, then $\lambda_5 = 0$, and we have $u'(\pi_n^*) = -1$. This violates Lemma 1. ■

9.2.1 Condition A holds: $\alpha \geq \frac{\theta}{1-\theta}$

When condition A holds, Lemma A7 shows that $\pi_m = \underline{\pi}$. This leads to the following lemma:

LEMMA 2. *Suppose that the magnitude of the shock α is large relative to its frequency θ, in the sense that*

$$\alpha \geq \frac{\theta}{1-\theta}. \tag{A}$$

Then (i.) all payoffs on the payoff frontier can be sustained by pure strategies; in other words, $\pi_0 = \underline{\pi}$. And (ii.), the payoff frontier is everywhere strictly downward sloping, that is, $u'(\pi) < 0$ for all $\pi \in [\underline{\pi}, \overline{\pi}]$. The payoff frontier is therefore maximized at $\underline{\pi}$, that is, $\pi_m = \underline{\pi}$.

Proof. Lemma A7 implies that $\pi_m = \underline{\pi}$ here. Since $\underline{\pi} \leq \pi_0 \leq \pi_m$, this implies that all three are equal, and gives (i). For (ii), given $\pi_m = \underline{\pi}$, we see that $u'(\pi) = -\frac{c'}{y'}$ for all π by Lemma A5 and Theorem 1. The only thing remaining to check for (ii) is then that $\underline{e} > 0$. Suppose to the contrary that $\underline{e} = 0$, then $u'(\underline{\pi}) = 0$ and the RC_S binds by Lemma A6. This implies that $\pi_s^*(\underline{\pi}) = \frac{1}{\delta}(\underline{\pi} + (1-\delta)(1+\theta\alpha)b_s) > \underline{\pi}$. However, as we see from Lemma 4 below (which does not rely on this lemma), $\pi_s^*(\underline{\pi}) \leq \underline{\pi}$ when $\alpha > \frac{\theta}{1-\theta}$. ■

Now to characterize the dynamics of the relationship, we first note the following two results.

LEMMA 4. *The optimal continuation profit in a period with a shock is given by*

$$\pi_s^*(\underline{\pi}) = \underline{\pi} \quad \textit{and} \quad \pi_s^*(\pi) < \pi \quad \textit{for all} \quad \pi > \underline{\pi}.$$

Proof. Combining the FOC of π_s and the envelop condition, we have that

$$1 + u'(\pi) = \theta(1 + u'(\pi_s)) + \lambda_2.$$

Note that if $\lambda_2 = 0$, we have $u'(\pi_s) > u'(\pi)$ since $1 + u'(\pi) > 0$ by Lemma 1. This implies that $\pi_s^*(\pi) < \pi$.

Now suppose $\lambda_2 > 0$. There are 2 possibilities. First, $\lambda_3 = 0$. In this case, we have

$$\theta u'(\pi_s) = -\frac{\theta - \alpha(1-\theta)}{1+\alpha} + \alpha\frac{1-\theta}{1+\alpha}u'(\pi).$$

And given $\alpha > \frac{\theta}{1-\theta}$, we can check that $u'(\pi_s) > u'(\pi)$, and again, we have $\pi_s^*(\pi) < \pi$.

Second, $\lambda_3 > 0$. In this case, we have $\pi_s^*(\pi) = \underline{\pi}$ (since this is the only way to have both $\lambda_2 > 0$ and $\lambda_3 > 0$). Therefore, $\pi_s^*(\pi) < \pi$ except when $\pi = \underline{\pi}$ and we have $\pi_s^*(\underline{\pi}) = \underline{\pi}$. ■

LEMMA 5. *Optimal effort $e^*(\pi)$ satisfies*

$$0 < e^*(\pi) < e_{fb} \text{ for all } \pi \in [\underline{\pi}, \overline{\pi}].$$

where e_{fb} is first best effort. Moreover, $e^*(\pi)$ is weakly increasing in π and satisfies

$$e^*(\pi_s^*(\pi)) < e^*(\pi) \quad \text{for all } \pi_s^*(\pi) < \pi.$$

Proof. The first part follows from Lemma 1 and Lemma A5 (which implies that $u' = -\frac{c'}{y'}$). The second part follows from Lemma 4 directly. ■

To further characterizes the dynamics, we need to distinguish two cases, i.e. whether Condition B is satisfied or not. To see the role that Condition B plays, note that the payoff frontier can be broadly classified into three regions: the right region ($\pi \in (\pi^R, \overline{\pi}]$ with $u'(\pi) > -\frac{1}{1+\alpha\theta}$), the middle region ($\pi \in [\pi^L, \pi^R]$ with $u'(\pi) = -\frac{1}{1+\alpha\theta}$), and the left region ($\pi \in [\underline{\pi}, \pi^L)$ with $< -\frac{1}{1+\alpha\theta}$). Condition B is satisfied if and only if $\pi_s^*(\pi^L) = \underline{\pi}$.

LEMMA 6. *Let $\underline{e} > 0$ denote the effort level for which $y(\underline{e}) = \underline{\pi}$. Then the optimal bonus in a shock state $b_s^*(\pi)$ is equal to zero for all $\pi \in [\underline{\pi}, \overline{\pi}]$ if and only if*

$$\frac{c'(\underline{e})}{y'(\underline{e})} \geq \frac{1 - \alpha(1-\theta)}{1+\theta\alpha}. \tag{B}$$

Proof. Suppose Condition B holds, which is equivalent to $u'(\underline{\pi}) \leq \frac{\alpha(1-\theta)-1}{1+\theta\alpha}$.We need to show that $b_s^*(\pi) = 0$ for all $\pi \in [\underline{\pi}, \overline{\pi}]$. Note that

$$\lambda_3 = \lambda_2 + \theta\alpha - (1+\theta\alpha)(1+u'(\pi)).$$

Since $\lambda_1 > 0$ (by Lemma 1) and $\lambda_2 \geq 0$ (by Kuhn-Tucker), it is immediate that $\lambda_3 > 0$ if $u'(\pi) < -\frac{1}{1+\theta\alpha}$. And, therefore, the complementary slackness condition implies that $b_s^*(\pi) = 0$.

Now consider $u'(\pi) > -\frac{1}{1+\theta\alpha}$. Suppose to the contrary that $b_s^*(\pi) > 0$. This implies that $\lambda_3 = 0$, and, thus,

$$\begin{aligned}
\theta u'(\pi_s) &= -\frac{\theta - \alpha(1-\theta)}{1+\alpha} + \alpha\frac{1-\theta}{1+\alpha}u'(\pi) \\
&> -\frac{\theta - \alpha(1-\theta)}{1+\alpha} - \alpha\frac{1-\theta}{1+\alpha}(\frac{1}{1+\theta\alpha}) \\
&= \theta\frac{\alpha(1-\theta)-1}{1+\theta\alpha} \\
&\geq \theta u'(\underline{\pi}),
\end{aligned}$$

where the last inequality follows from Condition B and Lemma A5. This is a contradiction since $\pi_s \geq \underline{\pi}$. This implies that $b_s^*(\pi) = 0$ for all $u'(\pi) > -\frac{1}{1+\theta\alpha}$.

Moreover, the above allows us showing that u is strictly concave in a neighborhood of $\underline{\pi}$. To see this, we first show that $\lambda_2 > 0$ in a neighborhood of $\underline{\pi}$. Suppose the contrary, we have a decreasing sequence of π_i converging to $\underline{\pi}$ such that $\lambda_2(\pi_i) = 0$. By Lemma 4, we have $\pi_s^*(\pi_i) < \pi_i$, and, thus, $\pi_s^*(\pi_i)$ also converge to $\underline{\pi}$. This implies that,

$$1 + u'(\underline{\pi}) = \lim_{i\to\infty}(1 + u'(\pi_i)) = \lim_{i\to\infty}\theta(1 + u'(\pi_s^*(\pi_i))) = \theta\left(1 + u'(\underline{\pi})\right),$$

so that $1 + u'(\underline{\pi}) = 0$, and this is a contradiction by Lemma 1. Now given that $\lambda_2 > 0$ in a neighborhood of $\underline{\pi}$, and that $b_s^*(\pi) = 0$, we must have $\pi_s^*(\pi) = \underline{\pi}$, and, from the $\mathrm{IC_N}$ constraint, we have

$$\pi = (1-\delta)y(e^*(\pi)) - \delta\underline{\pi}.$$

This implies that $y(e^*(\pi))$ is strictly increasing in π in a neighborhood of $\underline{\pi}$, and by Lemma A5, we have that u' is strictly increasing in a neighborhood of $\underline{\pi}$.

Now suppose $u'(\pi) = -\frac{1}{1+\theta\alpha}$. Again suppose to the contrary that $b_s^*(\pi) > 0$, and using the same argument as above, we will again have $u'(\pi_s) = u'(\underline{\pi})$, and, thus, by the strict concavity we have $\pi_s = \underline{\pi}$. Now consider all payoffs with $u'(\pi) = -\frac{1}{1+\theta\alpha}$, and let π_l be the left boundary of this set. For $\pi' < \pi_l$, we have $u'(\pi') > -\frac{1}{1+\theta\alpha}$ by definition. Then by above, we have $b_s^*(\pi') = 0$. Then upper-hemicontinuity of the optimizers imply that $b_s^*(\pi_l) = 0$ (this should read as that 0 is an element of equilibrium bonus associated with π_l). Then $\mathrm{IC_N}$ constraint implies that

$$\pi_l = (1-\delta)y(e^*(\pi_l)) - \delta\underline{\pi}.$$

But for all payoffs with $u'(\pi) = -\frac{1}{1+\theta\alpha}$, From $\mathrm{IC_N}$ constraint, we have

$$\begin{aligned}(1-\delta)y(e^*(\pi_l)) - \delta\underline{\pi} &= \pi_l \\ &\leq \pi + (1-\delta)(1+\theta\alpha)b_s^*(\pi) \\ &= (1-\delta)y(e^*(\pi)) - \delta\underline{\pi}.\end{aligned}$$

And since $e^*(\pi_l) = e^*(\pi)$, the inequality is an equality, we must then have $b_s^*(\pi) = 0$, which is a contradiction. This concludes the proof that $b_s^*(\pi) = 0$ all $\pi \in [\underline{\pi}, \overline{\pi}]$.

Now consider Condition B fails. In this case, $u'(\underline{\pi}) > \frac{\alpha(1-\theta)-1}{1+\theta\alpha}$. Suppose to the contrary that $b_s^*(\pi) = 0$ for all $\pi \in [\underline{\pi}, \overline{\pi}]$. By $\mathrm{IC_S}$, this implies that

$$\pi - (1-\delta)y(e^*(\pi)) = \delta\pi_s^*(\pi).$$

Since for each π, there is a unique $e^*(\pi)$ (which follows because $\frac{c'}{y'}$ is strictly increasing), the above implies that $\pi_s^*(\pi)$ is unique, and thus, π_s^* is continuous.

Note that if $u'(\pi) = -\frac{1}{1+\theta\alpha}$, we then have

$$\theta(1 + u'(\pi_s^*(\pi))) = (1 + u'(\pi)) - \lambda_2 \leq \frac{\theta\alpha}{1 + \theta\alpha}$$

This implies that $u'(\pi_s^*(\pi)) \leq \frac{\alpha(1-\theta)-1}{1+\theta\alpha} < u'(\underline{\pi})$. Therefore, $\pi_s^*(\pi) > \underline{\pi}$.

Consider a π' such that $u'(\pi') = -\frac{1}{1+\theta\alpha} + \varepsilon$ for a small enough ε. Since

$$\lambda_3(\pi') = \lambda_2(\pi') + \theta\alpha - (1 + \theta\alpha)(1 + u'(\pi')),$$

and $\lambda_3 \geq 0$, we must have $\lambda_2 > 0$. Given $b_s^*(\pi') = 0$ and the complementarity slackness of $\mathrm{RC_S}$, we then must have $\pi_s^*(\pi') = \underline{\pi}$. Therefore, $u'(\pi_s^*(\pi')) = u'(\underline{\pi}) > \frac{\alpha(1-\theta)-1}{1+\theta\alpha}$.

Now takes an increasing sequence of π_i' such that π_i' converges to π where recall that $u'(\pi) = -\frac{1}{1+\theta\alpha}$. Note that $\pi_s^*(\pi) > \underline{\pi}$ but $\pi_s^*(\pi_i') = \underline{\pi}$, and this violates the continuity of π_s^*. ■

LEMMA 7. *Suppose that Condition B holds. Then the optimal bonus in a period without a shock is given by*

$$b_n^*(\pi) = \frac{\delta}{1-\delta}\left(\overline{\pi} - \pi_s^*(\pi)\right).$$

Proof. This follows directly from $b_s^*(\pi) = 0$ (Lemma 6) and $\pi_n^*(\pi) = \overline{\pi}$ (Lemma 3). ■

LEMMA 8. *Suppose that Condition B does not hold. Then there exist three profit levels π_1, π_2, π_3 that satisfy $\underline{\pi} \leq \pi_1 < \pi_2 < \pi_3 < \overline{\pi}$ such that (i.) the optimal bonus $b_s^*(\pi)$ that the firm pays in a shock period is strictly positive if and only if $\pi \in (\pi_1, \pi_3)$. And, (ii.), when $b_s^*(\pi)$ is strictly positive it is non-monotonic in π; in particular*

$$\frac{\mathrm{d}b_s^*(\pi)}{\mathrm{d}\pi} > 0 \quad \textit{for all } \pi \in (\pi_1, \pi_2) \quad \textit{and} \quad \frac{\mathrm{d}b_s^*(\pi)}{\mathrm{d}\pi} < 0 \quad \textit{for all } \pi \in (\pi_2, \pi_3).$$

Proof. Combining the envelop condition and the FOC wrt b_s, we have

$$\lambda_3 = \lambda_2 + \theta\alpha - (1 + \theta\alpha)(1 + u'(\pi)).$$

This implies that if $u'(\pi) < -\frac{1}{1+\theta\alpha}$, we have $\lambda_3 > 0$, and, thus, $b_s^*(\pi) = 0$. Let $\pi_3 = \inf\{\pi, u'(\pi) < -\frac{1}{1+\theta\alpha}\}$.

Now consider $u'(\pi) \geq -\frac{1}{1+\theta\alpha}$. From the the FOC wrt π_s, we have

$$\theta u'(\pi_s) = \frac{\alpha(1-\theta)-\theta}{1+\alpha} + \alpha\frac{1-\theta}{1+\alpha}u'(\pi) - \lambda_3.$$

Since $u'(\pi_s) \leq u'(\underline{\pi})$, this implies that

$$u'(\pi) \leq \frac{1+\alpha}{\alpha(1-\theta)}(\theta u'(\underline{\pi}) - \frac{\alpha(1-\theta)-\theta}{1+\alpha} + \lambda_3).$$

Define π_1 as the (smallest) payoff such that $u'(\pi_1) = \frac{1+\alpha}{\alpha(1-\theta)}(\theta u'(\underline{\pi}) - \frac{\alpha(1-\theta)-\theta}{1+\alpha})$. The above then implies that for $\pi < \pi_1$, we have $\lambda_3(\pi) > 0$, and, thus, $b_s^*(\pi) = 0$. Note that Condition A implies that $u'(\pi_1) < u'(\underline{\pi})$ (so that $\pi_1 > \underline{\pi}$). Condition B implies that $u'(\pi_1) > -\frac{1}{1+\theta\alpha}$. Therefore, Lemma A6 implies that for $\pi < \pi_1$, the $\mathrm{RC_S}$ is binding. When both $\mathrm{RC_S}$ is binding and $b_s^*(\pi)$ is equal to 0, we then also have $\pi_s^*(\pi) = \underline{\pi}$. By $\mathrm{IC_N}$, it is then clear that $y(e(\pi))$ is strictly increasing in $\pi \in [\underline{\pi}, \pi_1]$, and, thus, u is strictly concave in this region.

Now define $\pi_2 = \sup\{\pi, u'(\pi) > -\frac{1}{1+\theta\alpha}\}$. Now for $\pi \in (\pi_1, \pi_2)$, we show that we must have $\lambda_3(\pi) = 0$. Now suppose the contrary that $\lambda_3 > 0$, and we have $b_s^*(\pi) = 0$. Then on the one hand, by the definition of π_1, we must then have

$$\begin{aligned}
\theta u'(\pi_s^*(\pi)) &= \frac{\alpha(1-\theta)-\theta}{1+\alpha} + \alpha\frac{1-\theta}{1+\alpha}u'(\pi) - \lambda_3 \\
&< \frac{\alpha(1-\theta)-\theta}{1+\alpha} + \alpha\frac{1-\theta}{1+\alpha}u'(\pi_1) \\
&= \theta u'(\underline{\pi}).
\end{aligned}$$

And therefore, $\pi_s^*(\pi) > \underline{\pi}$. On the other hand, the definition of π_2 implies that $\mathrm{RC_S}$ binds at π. But this is a contradiction because we have just shown that $\pi_s^*(\pi) > \underline{\pi}$ and $b_s^*(\pi) = 0$.

Note that we have just shown that for $\pi \in (\pi_1, \pi_2)$, $\lambda_3(\pi) = 0$. Therefore, we have

$$\theta u'(\pi_s^*(\pi)) = \frac{\alpha(1-\theta)-\theta}{1+\alpha} + \alpha\frac{1-\theta}{1+\alpha}u'(\pi).$$

Using the result above that u is strictly concave in $[\underline{\pi}, \pi_1]$ (and that $u'(\pi_s^*(\pi)) < u'(\pi)$), we can then show that u is strictly concave in $\pi \in (\pi_1, \pi_2)$. This then implies that $\pi_s^*(\pi)$ is strictly increasing in π. Since $\mathrm{RC_S}$ is binding for $\pi \in (\pi_1, \pi_2)$, so that $(1-\delta)(1+\alpha)b_s^*(\pi) = \delta(\pi_s^*(\pi) - \underline{\pi})$, we must then have $\frac{\mathrm{d}b_s^*(\pi)}{\mathrm{d}\pi} > 0$ *for all* $\pi \in (\pi_1, \pi_2)$.

Finally, for $\pi \in (\pi_2, \pi_3)$, we have $u'(\pi) = -\frac{1}{1+\theta\alpha}$, and thus, $u'(\pi_s^*(\pi)) = \frac{\alpha(1-\theta)-1}{1+\theta\alpha} > -\frac{1}{1+\theta\alpha}$. This implies that $\pi_s^*(\pi)$ is unique (since u is strictly concave for $\pi < \pi_2$). Therefore, for all $\pi \in (\pi_2, \pi_3)$, $\pi_s^*(\pi)$ is identical. Using $\mathrm{IC_N}$, we then get that $\frac{\mathrm{d}b_s^*(\pi)}{\mathrm{d}\pi} < 0$ *for all* $\pi \in (\pi_2, \pi_3)$. ■

LEMMA 9. *Suppose that Condition B does not hold. Then the optimal bonus in a no shock period is given by*

$$b_n^*(\pi) = b_s^*(\pi) + \frac{\delta}{1-\delta}\left(\overline{\pi} - \pi_s^*(\pi)\right).$$

Proof. This is again directly the consequence of $\pi_n^*(\pi) = \overline{\pi}$. ■

9.2.2 Condition A Fails $\alpha < \frac{\theta}{1-\theta}$

In this case, the payoff frontier and the dynamics is very similar to the previous, although it is no longer true that we always have $\pi_s^*(\pi) \leq \pi$. To see this, note that for $\pi \in (\pi_1, \pi_2)$ (in Lemma 8), we have

$$\theta u'(\pi_s^*(\pi)) = \frac{\alpha(1-\theta)-\theta}{1+\alpha} + \alpha\frac{1-\theta}{1+\alpha}u'(\pi).$$

And for $u'(\pi) \leq 0$, we always have $u'(\pi_s^*(\pi)) \geq u'(\pi)$ when condition A is satisfied.

When Condition A fails, it is no longer true that we always have $u'(\pi_s^*(\pi)) \geq u'(\pi)$. This is true if and only if

$$u'(\pi) \leq \frac{\alpha-(1+\alpha)\theta}{(1+\alpha)\theta-\alpha(1-\theta)}.$$

This observation implies that the dynamics of the relationship can be divided into two cases, depending on the size of $u'(\underline{\pi})$.

Case 1: $u'(\underline{\pi}) \leq \frac{\alpha-(1+\alpha)\theta}{(1+\alpha)\theta-\alpha(1-\theta)}$ In this case, we again always have $\pi_s^*(\pi) \leq \pi$, and the dynamics of the relationship is identical when Condition A holds. Specifically, if Condition B holds, the dynamics is characterized by that in Proposition 1. Otherwise, the dynamics is given by Proposition 2.

Case 2: $u'(\underline{\pi}) > \frac{\alpha-(1+\alpha)\theta}{(1+\alpha)\theta-\alpha(1-\theta)}$ In this case, it is no longer true that we always have $\pi_s^*(\pi) \leq \pi$. Define π_α as the payoff such that

$$u'(\pi_\alpha) = \frac{\alpha-(1+\alpha)\theta}{(1+\alpha)\theta-\alpha(1-\theta)}.$$

Note that given $\alpha < \frac{\theta}{1-\theta}$, we have $u'(\pi_\alpha) < 0$, so $\pi_\alpha > \pi_m$. In addition, we can check that $u'(\pi_\alpha) > -\frac{1}{1+\alpha\theta}$. Now let $\pi^L = \sup\{\pi : u'(\pi) > -\frac{1}{1+\alpha\theta}\}$ and $\pi^R = \inf\{\pi : u'(\pi) < -\frac{1}{1+\alpha\theta}\}$. We have $\pi_m < \pi_\alpha < \pi^L \leq \pi^R < \overline{\pi}$. Note that π^L corresponds to π_2 (in Lemma 9) and π^R corresponds to π_3. The dynamics of the relational contract can be summarized by what happens in the subintervals. Note that in all these intervals we have $\frac{c'}{y'} = -u'(\pi)$ and $\pi_n^*(\pi) = \overline{\pi}$.

(i): For $\pi \in [\pi_m, \pi_\alpha]$, we have

$$\begin{aligned}\theta u'\left(\pi_s^*\left(\pi\right)\right) &= \frac{\alpha\left(1-\theta\right)-\theta}{1+\alpha}+\alpha\frac{1-\theta}{1+\alpha}u'\left(\pi\right);\\ (1-\delta)b_s^*\left(\pi\right) &= \delta(\pi_s^*\left(\pi\right)-\underline{\pi}].\end{aligned}$$

In this region, we have $\pi_s^*\left(\pi\right) \geq \pi$ and the inequality is strict for all $\pi < \pi_\alpha$. Therefore, the relationship always improves in this region.

(ii): For $\pi \in (\pi_\alpha, \pi^L)$, we have

$$\begin{aligned}\theta u'\left(\pi_s^*\left(\pi\right)\right) &= \frac{\alpha\left(1-\theta\right)-\theta}{1+\alpha}+\alpha\frac{1-\theta}{1+\alpha}u'\left(\pi\right);\\ (1-\delta)b_s^*\left(\pi\right) &= \delta(\pi_s^*\left(\pi\right)-\underline{\pi}].\end{aligned}$$

- Here, however, $\pi_s^*\left(\pi\right) < \pi$ and the relationship suffers when the shock state occurs. Note that $\mathrm{RC_S}$ binds in this region and we have $\frac{\mathrm{d}b_s^*(\pi)}{\mathrm{d}\pi} > 0$.

(iii): For $\pi \in [\pi^L, \pi^R]$,

$$\begin{aligned}u'\left(\pi_s^*\left(\pi\right)\right) &= \frac{\alpha\left(1-\theta\right)-1}{1+\theta\alpha}\\ \left(1+\theta\alpha\right)b_s^*\left(\pi\right) &= y\left(e^*(\pi)\right)+\frac{\delta}{1-\delta}(\pi_s^*\left(\pi\right)-\pi).\end{aligned}$$

Here, we again have $\pi_s^*\left(\pi\right) < \pi$. Note that $\mathrm{RC_S}$ is slack in this region and we have $\frac{\mathrm{d}b_s^*(\pi)}{\mathrm{d}\pi} < 0$.

(iv): For $\pi \in [\pi^R, \overline{\pi}]$,

$$\begin{aligned}\theta(u'\left(\pi_s^*\left(\pi\right)\right)+1) &= u'\left(\pi\right)+1;\\ b_s^*\left(\pi\right) &= 0.\end{aligned}$$

9.3 Long-term Contract

Recall that $h^t = \{y_1, ...y_t\}$ is the history of past outputs. Let $m^t = \{m_1, ..m_t\}$ be the history of past announcements, and t_n be the last time the firm announces no shock (and $t_n = 0$ if the firm has never announced no shock.) Let $b_t(h^t, m^t)$ be the firm's payment to the worker in period t.

Proposition LT1: As T approaches ∞, the following class of contracts approaches first best.

$$b_t(h^t, m^t) = \begin{cases} 0 & \text{if } h^t \neq \{y^{FB}, ..., y^{FB}\} \\ 0 & \text{if } m_t = n \text{ and } t < t_n + T \\ (c(e^{FB}) + \underline{u})(1 + \delta^{-1} + \delta^{-2} + ... + \delta^{-(t-1-t_n)}) & \textit{otherwise.} \end{cases}$$

Proof. To simplify the exposition, we normalize $\underline{u}$ to be zero. Under the construction above, the worker's payoff is 0 by putting e^{FB} in each period. Any other effort choice (except $e = 0$) gives the worker a negative payoff. Therefore, choosing $e_t = e^{FB}$ along the equilibrium path is a best response for the worker.

Given that the worker always chooses $e_t = e^{FB}$ (so that the worker's payoff is always 0), the firm's payoff is equal to the value of the relationship. This implies that the firm's payoff is maximized if the value of the relationship is maximized. Alternatively, this implies that the firm will choose a strategy that minimizes the destruction of the value of the relationship. Note that the value of the relationship is destroyed when the firm pays out the bonus to the worker in a shock state.

The firm chooses his strategy to minimize the value destruction. The strategy of the firm is a mapping from his private history of

For the firm, his value that minimizes the surplus destruction (from paying out money when it is inefficient to do so.)

It is clear that if the state is no-shock, the firm would like to announce that it is the no-shock state because this allows for paying the worker cheaply and thus avoids surplus destruction.

It remains to check that when it is a shock state, the firm will be truthtelling and no to claim that it is a no-shock state and pays out money. Given that the contract has the feature of renewal (after a no-shock state is announced), it suffices to consider the following strategy: the firm make truthful announcements until period n (in which he always announces no shock has happened). Given this strategy, the expected destruction of surplus can be calculated explicitly. In particular, let $k = \alpha c(e^{FB})$ and V_n be the expected destruction of surplus when there no-shock has been announced for the past n periods. (The expected destruction of surplus is given by V_0). Then

$$\begin{aligned} V_0 &= (1-\theta)(0+\delta V_0) + \theta\delta V_1; \\ V_1 &= (1-\theta)(0+\delta V_0) + \theta\delta V_2; \\ &\dots \\ V_{n-1} &= (1-\theta)(0+\delta V_0) + \theta\delta V_n, \end{aligned}$$

and

$$V_n = \theta k(1 + \delta^{-1} + \delta^{-2} + \dots + \delta^{-(n-1)}) + \delta V_0.$$

Or alternatively,

$$V_n = \theta k \frac{1-\delta^n}{\delta^{(n-1)}(1-\delta)} + \delta V_0$$

From the first $n-1$ equations, we see that for $m<n$,

$$V_m - \theta\delta V_{m+1} = (1-\theta)\delta V_0,$$

and this implies that

$$V_0 - (\theta\delta)^n V_n = (1-\theta)\delta\frac{1-(\theta\delta)^n}{1-\theta\delta}V_0,$$

or

$$V_n = \frac{1-\delta+\delta(\theta\delta)^n(1-\theta)}{(\theta\delta)^n(1-\theta\delta)}V_0$$

Now using $V_n = \theta k\frac{1-\delta^n}{\delta^{(n-1)}(1-\delta)} + \delta V_0$, we have

$$\begin{aligned} V_0 &= \frac{\theta k\frac{1-\delta^n}{\delta^{(n-1)}(1-\delta)}}{\frac{1-\delta+\delta(\theta\delta)^n(1-\theta)}{(\theta\delta)^n(1-\theta\delta)}-\delta} \\ &= \delta\theta k\frac{\theta^n(1-\delta^n)}{(1-\delta)^2}\frac{(1-\theta\delta)}{(1-(\theta\delta)^{n+1})}. \end{aligned}$$

Now we want to show that this term is minimized at $n=T$.

Note that

$$\begin{aligned} &\frac{\theta^{n-1}(1-\delta^{n-1})}{1-(\theta\delta)^n} - \frac{\theta^n(1-\delta^n)}{1-(\theta\delta)^{n+1}} \\ =\ &\frac{\theta^{n-1}}{(1-(\theta\delta)^n)(1-(\theta\delta)^{n+1})}((1-\delta^{n-1})(1-(\theta\delta)^{n+1}) - (1-(\theta\delta)^n)\theta(1-\delta^n)) \\ =\ &\frac{\theta^{n-1}}{(1-(\theta\delta)^n)(1-(\theta\delta)^{n+1})}\left(1-\delta^{n-1}-\theta+\theta\delta^n-(\theta\delta)^{n+1}+\theta(\theta\delta)^n\right) \\ >\ &\frac{\theta^{n-1}}{(1-(\theta\delta)^n)(1-(\theta\delta)^{n+1})}\left(1-\delta^{n-1}-\theta\right). \end{aligned}$$

Therefore, for

$$n > 1+\frac{\log(1-\theta)}{\log\delta},$$

we have $\frac{\theta^{n-1}(1-\delta^{n-1})}{1-(\theta\delta)^n} - \frac{\theta^n(1-\delta^n)}{1-(\theta\delta)^{n+1}} > 0$, and thus $\frac{\theta^n(1-\delta^n)}{1-(\theta\delta)^{n+1}}$ is decreasing with respect to $n > 1+\frac{\log(1-\theta)}{\log\delta}$.

Let M be

$$\min_{n\le 1+\frac{\log(1-\theta)}{\log\delta}}\{\delta\theta k\frac{\theta^n(1-\delta^n)}{(1-\delta)^2}\frac{(1-\theta\delta)}{(1-(\theta\delta)^{n+1})}\}.$$

Therefore, for $n>M$, $\delta\theta k\frac{\theta^n(1-\delta^n)}{(1-\delta)^2}\frac{(1-\theta\delta)}{(1-(\theta\delta)^{n+1})}$ is decreasing in n, and moreover, $\delta\theta k\frac{\theta^n(1-\delta^n)}{(1-\delta)^2}\frac{(1-\theta\delta)}{(1-(\theta\delta)^{n+1})}$ approaches 0 as n goes to infinity. Therefore, there exists an $N>M$ such that for $n>N$,

$$\delta\theta k\frac{\theta^n(1-\delta^n)}{(1-\delta)^2}\frac{(1-\theta\delta)}{(1-(\theta\delta)^{n+1})} < M.$$

Now take any $T > N$, we see that $\delta\theta k \frac{\theta^n(1-\delta^n)}{(1-\delta)^2}\frac{(1-\theta\delta)}{(1-(\theta\delta)^{n+1})}$ is minimized at $n = T$.

Finally, the expected surplus that gets destroyed is

$$\delta\theta k \frac{\theta^T(1-\delta^T)}{(1-\delta)^2}\frac{(1-\theta\delta)}{(1-(\theta\delta)^{T+1})},$$

so as T goes to infinity, this contract approximates first best. ■

9.4 Liquidity Constraint

In this case, we have the extra liquidity constraint that

$$b_n \leq (1+m)y$$

for some $m \geq 0$. As can be seen below, this constraint significantly complicates the problem. Part of the reason is that the payoff frontier is no longer differentiable. To make the analysis more tractable, we assume that it is impossible for the firm to pay the worker in a shock state ($\alpha = \infty$), so that $b_s = 0$.

Using the same proof as in the unconstrained case, it can be shown that $\mathrm{IC_N}$ binds, so

$$\delta\left(\pi_n - \pi_s\right) = \left(1-\delta\right) b_n.$$

This allows us substituting out b_n and rewriting the liquidity constraint as

$$\delta\pi_n \leq \pi + (1-\delta)my. \tag{Liq}$$

Similar to the unconstrained case, we can show that there exists a π_0 such that for all $\pi \geq \pi_0$ the payoff frontier $u(\pi)$ is sustained by pure strategy. And for $\pi < \pi_0$ the payoff frontier $u(\pi)$ is sustained by randomization. To the right of π_0, the payoff frontier satisfies the following functional equation:

$$\pi + u(\pi) = \max_{e,\pi_s,\pi_n} \left(1-\delta\right)\left(y\left(e\right) - c\left(e\right)\right) + \theta\delta\left(\pi_s + u\left(\pi_s\right)\right) + \left(1-\theta\right)\delta\left(\pi_n + u\left(\pi_n\right)\right) \tag{3}$$

such that

$$\pi = \left(1-\delta\right) y\left(e\right) + \delta\pi_s. \tag{$\mathrm{IC_N}$}$$

$$\delta\pi_n \leq \pi + (1-\delta)my(e). \tag{Liquidity}$$

$$e \geq 0. \tag{$\mathrm{Neg_S}$}$$

$$\pi_n \leq \overline{\pi}. \tag{Self-enf}$$

Unlike the unconstrained case, the payoff frontier is longer differentiable so we cannot use the Lagrangian method here. Nevertheless, the concavity of u implies that at each point both the left and the right derivative exist. Similar to the unconstrained case, we again have $u'_{-}(\pi) > -1$ for all π, where $u'_{-}(\pi)$ stands for the left derivative of u at π. This implies that

$$\pi_n = \min\{\overline{\pi}, \frac{1}{\delta}(\pi + (1-\delta)my)\}.$$

9.4.1 Structure of the Payoff Frontier

To proceed with our analysis, we first show that for $\pi \geq \pi_0$, the payoff frontier can be classified into (at most) three regions. The right region is the same as in the unconstrained case so that the liquidity constraint is slack. The left region is one in which the liquidity binds and that $\pi_n < \overline{\pi}$. In other words, the relationship does not jump back to $\overline{\pi}$ after a no shock state. In the middle region, both the liquidity constraint binds and that $\pi_n = \overline{\pi}$.

Our lemma below shows that there exists a threshold π_r such that if the liquidity constraint is slack if and only if $\pi > \pi_r$.

Lemma L1: *There exists $\pi_{l.}$ and π_r with $\pi_0 \leq \pi_{l.} \leq \pi_r < \overline{\pi}$ such that the following holds. (i) If $\pi > \pi_r$, $\pi_n = \overline{\pi}$ and $\pi + (1-\delta)my > \delta\overline{\pi}$. (ii) If $\pi \in [\pi_{l.}, \pi_r]$, $\pi_n = \overline{\pi}$ and $\pi + (1-\delta)my = \delta\overline{\pi}$. (iii) If $\pi < \pi_{l.}$, $\pi_n < \overline{\pi}$ and $\pi + (1-\delta)my = \delta\overline{\pi}$.*

Proof. To prove the existence of π_r, we show that if $\pi + (1-\delta)my > \delta\overline{\pi}$, then for all $\pi' \geq \pi$, we also have $\pi' + (1-\delta)my' > \delta\overline{\pi}$.

Consider two cases. In Case 1, we have $e(\pi) = 0$. In this case, we have

$$\pi > \delta\overline{\pi}.$$

Now since $\pi' > \pi$ and $y' \geq 0$, the result is immediate.

In Case 2, we have $e(\pi) > 0$. Then the same argument as in the unconstrained case (Lemma A5) shows that u is differentiable at π, and specifically,

$$u'(\pi) = -\frac{c'}{y'}.$$

For $\pi' > \pi$, the same argument as in the unconstrained case shows that

$$\frac{c'}{y'}\left(\pi'\right) \geq -u'_{+}(\pi') \geq -u'_{+}(\pi) = \frac{c'}{y'}(\pi).$$

This implies that $e(\pi') \geq e(\pi)$, and again the liquidity constraint is slack at π'. Finally, note that $\pi_r < \overline{\pi}$ because for all $\pi > \delta\overline{\pi}$, we have $\pi + (1-\delta)my > \delta\overline{\pi}$.

To prove the existence of the middle region (and thus π_l), we show that if $\pi_n(\pi) = \overline{\pi}$, then for all $\pi' > \pi$, we have $\pi_n(\pi') = \overline{\pi}$. Suppose the contrary, then there exists $\pi' > \pi$ such that

$$\pi'_n = \pi' + (1-\delta)my' < \pi + (1-\delta)my = \overline{\pi}.$$

Note the above implies that $e(\pi) > 0$. Now as we move from π to $\pi - \varepsilon$ for some small $\varepsilon > 0$, we see that by decreasing e and π_n (which is possible because $e > 0$) and keeping π_s the same, we have

$$\frac{c'}{y'}(\pi) \leq (m+1)(1-\theta)(1+u'_{-}(\overline{\pi})) - u'_{-}(\pi).$$

Similarly, as we move from π' to $\pi' + \varepsilon$ for some small $\varepsilon > 0$, we see that by increasing e' and π'_n (which is possible because $\pi'_n < \overline{\pi}$) and keeping π'_s the same, we have

$$\frac{c'}{y'}(\pi') \geq (m+1)(1-\theta)(1+u'_{+}(\pi_n(\pi')) - u'_{+}(\pi'),$$

Since $\pi' > \pi$, we have

$$-u'_{+}(\pi') \geq -u'_{-}(\pi).$$

Since $\pi_n(\pi') < \overline{\pi}$, we have

$$u'_{+}(\pi_n(\pi')) \geq u'_{-}(\overline{\pi}).$$

Combining the above, we have

$$\frac{c'}{y'}(\pi') \geq \frac{c'}{y'}(\pi),$$

i.e., $e(\pi') \geq e(\pi)$.

But this contradicts that

$$\pi' + (1-\delta)(m)y' < \pi + (1-\delta)(m)y.$$

■

Lemma L1 implies that the right region always exists since $\pi_r < \overline{\pi}$. In contrast, the middle region or the left region does not always exist. This can occur, for example, when m is large and when $\underline{\pi}$ is large (meaning that y is large). In this case, the liquidity constraint never binds and we go back to the unconstrained case. At the end of this section, we give sufficient conditions of when the left and the middle region exist.

9.4.2 Properties of the Three Region

Before summarizing the properties of e, π_s, and π_n in the three regions, we first note that the maximizers are all unique.

Lemma L2: *For each π, there is a unique set of $e(\pi), \pi_s(\pi)$, and $\pi_n(\pi)$ that maximizes $u(\pi)$.*

Proof. Let $e_i, \pi_{s_i}, \pi_{n_i}$, $i = 1, 2$ be the associated effort and continuation payoffs as maximizers. Then for some $\alpha \in (0, 1)$, let

$$\begin{aligned} \pi_s &= \alpha \pi_{s_1} + (1-\alpha)\pi_{s_2}; \\ \pi_n &= \alpha \pi_{n_1} + (1-\alpha)\pi_{n_2}; \end{aligned}$$

Let e be the (unique) effort level such that

$$y(e) = \alpha y(e_1) + (1-\alpha) y(e_2).$$

Note that the concavity of y implies that $e \leq \alpha e_1 + (1-\alpha) e_2$.

It is clear that e, π_s, π_n is also a feasible solution. In addition, the concavity of u and convexity of e implies that this new set of choice is also a maximizer. Moreover, the strict concavity of y and the strict convexity of c implies that the value generated by this new set of choices is strictly larger than those from $e_i, \pi_{s_i}, \pi_{n_i}$, $i = 1, 2$ when $e_1 \neq e_2$.

Therefore, we must $e_1 = e_2$. It follows that $\pi_{s1} = \pi_{s2}$ from the promise-keeping constraint that $(\pi = (1-\delta)y + \delta \pi_s)$. Finally,

$$\pi_n = \min\{\overline{\pi}, \frac{1}{\delta}(\pi + (1-\delta)(m)y)\}$$

is also unique. ■

Since e, π_s, and π_n is upper-hemicontinuous in π, a direct consequence of the lemma above is that they are continuous in π.

Proposition L1: The set of effort and continuation payoffs ($e(\pi), \pi_s(\pi)$, *and* $\pi_n(\pi)$) satisfy the following.

(i): For $\pi > \pi_r$, the payoff frontier is differentiable, and

$$\begin{aligned} \frac{c'}{y'} &= -u'(\pi); \\ -(1-\theta) + \theta u'_+(\pi_s) &\leq u'(\pi) \leq -(1-\theta) + \theta u'_-(\pi_s); \\ \pi_n &= \overline{\pi}. \end{aligned}$$

Both e and $\pi_s$weakly increase with π.

(ii): For $\pi \in [\pi_{l.}, \pi_r]$, if $m \neq 0$,

$$\begin{aligned} y &= \frac{\delta\overline{\pi} - \pi}{(1-\delta)m}; \\ \pi_s &= \frac{(m+1)\pi + \delta\overline{\pi}}{m}; \\ \pi_n &= \overline{\pi}. \end{aligned}$$

e decreases with π and π_s increases with π.

If $m = 0$, we have $\pi_{l.} = \pi_r = \delta\overline{\pi}$, u is not differentiable at this point, and e and π_s satisfies

$$\begin{aligned} -u'_+(\pi_l) &\leq \frac{c'}{y'} \leq -u'_-(\pi_l); \\ -(1-\theta) + \theta u'_+(\pi_s) &\leq u'(\pi) \leq -(1-\theta) + \theta u'_-(\pi_s). \end{aligned}$$

(iii) For $\pi \in [\pi_{0.}, \pi_l]$,

$$(1+m)(1-\theta)(1+u'_+(\pi_n)) - \frac{c'}{y'} \leq u'_+(\pi) \leq u'_-(\pi) \leq (1+m)(1-\theta)(1+u'_-(\pi_n)) - \frac{c'}{y'}. \quad \text{(L-e-n)}$$

And if $\pi_s > \underline{\pi}$,

$$\begin{aligned} -(1+m)(1-\theta) + (1+m)\theta u'_+(\pi_s) + \frac{c'}{y'} &\leq mu'_+(\pi) \leq mu'_-(\pi) \quad \text{(L-e-s)} \\ &\leq -(1+m)(1-\theta) + (1+m)\theta u'_-(\pi_s) + \frac{c'}{y'}. \end{aligned}$$

In addition,

$$\theta u'_+(\pi_s) + (1-\theta) u'_+(\pi_n) \leq u'_+(\pi) \leq u'_-(\pi) \leq \theta u'_-(\pi_s) + (1-\theta) u'_-(\pi_n). \quad \text{(L-s-n)}$$

π_s is weakly increasing in π.

Proof. The inequalities in Lemma L3 are all equalities if u is differentiable. In this case, the equalities can be obtained directly from the Kuhn-Tucker conditions of Lagrangian associated the constrained maximization problem. The formal proof of the inequalities is mostly routine and is omitted here. We mention only two points here. First,when $m = 0$, the middle region consists of one point only: $\pi = \delta\overline{\pi}$. To see that u is not differentiable at $\delta\overline{\pi}$, note that from the right region, we have that

$$u'_+(\delta\overline{\pi}) = -\frac{c'}{y'}.$$

From the left region, we have that

$$u'_-(\delta\overline{\pi}) \geq -\frac{c'}{y'} + (1+m)(1-\theta)(1+u'_+(\pi_n)).$$

Since $u'_+(\pi_n) > -1$ as in the unconstrained case, we have that

$$u'_-(\delta\overline{\pi}) > u'_+(\delta\overline{\pi}),$$

and therefore, u is not differentiable at $\delta\overline{\pi}$.

Second, π_s is weakly increasing for $\pi \in [\pi_{0.}, \pi_l]$. To see this, we may assume that u is differentiable at π and π_s, and the argument can be adapted to the non-differentiable case. When u is differentiable at π and π_s, we can write L-e-s as

$$-(1+m)(1-\theta) + (1+m)\theta u'(\pi_s) + \frac{c'}{y'} = mu'(\pi)$$

Now as π increases, the right hand side weakly decreases. Now if π_s decreases, then $u'(\pi_s)$ weakly increases. Moreover, if π increases and π_s decreases, this implies that e increases. Consequently, $\frac{c'}{y'}$ strictly increases. In other words, if π_s decreases, the left hand side strictly increases. This leads to a contradiction. Therefore, we have that π_s increases weakly with π. ■

Note that $\pi_s(\pi)$ weakly increases with π in all three regions. Since $\pi_s(\pi)$ is continuous in π this implies that $\pi_s(\pi)$ weakly increases with π for all $\pi \in [\pi_{0.}, \overline{\pi}]$. In contrast, $e(\pi)$ is decreasing in the middle region. In other words, the worker's effort level increases as the firm's payoff decreases.

9.4.3 Dynamics

Here, we show that for all $\pi \in (\underline{\pi}, \overline{\pi})$, we have $\pi_s < \pi < \pi_n$. It is easy to show that $\pi_s < \pi < \pi_n$he technical difficulty is to rule out cases in which u has a line segment that contains all π_s, π, and π_n. We proceed with the following sequence of lemmas.

Lemma L3: If u is a line segment in $[\underline{\pi}, \pi_+]$, then $\pi_s < \pi < \pi_n$ for all $\pi \in (\underline{\pi}, \pi_+]$.

Proof. Note that $\pi_n(\underline{\pi}) > \underline{\pi}$ (because otherwise we have $\pi_n = \pi_s = \underline{\pi}$). Consider the following two cases. In case 1, $\pi_s(\pi) = \underline{\pi}$ for all $\pi \in (\underline{\pi}, \pi_+]$. In this case, $\pi = \delta\underline{\pi} + (1-\delta)y$, so that y is weakly increasing in π. This implies that $\pi_n = \min\{\overline{\pi}, \frac{1}{\delta}(\pi + (1-\delta)my)\}$ is strictly larger than π since $\frac{d\pi_n}{d\pi} > 1$ whenever $\pi_n < \overline{\pi}$. Therefore, the claim holds in this case.

In case 2, by the monotonicity of π_s in π, there exists a $\pi' < \pi_+$ such that $\pi_s(\pi) > \underline{\pi}$ if and only if $\pi > \pi'$. Now note that for $\pi \in (\pi', \pi_+]$, we must have $\pi_n \leq \pi_+$. This is because otherwise, the slope (left derivative) at π_n is strictly larger than that at π and the slope at π_n is weakly larger than that at π. This violates L-s-n.

In particular, the above implies that $\pi_s(\pi_+) \leq \pi_n(\pi_+) \leq \pi_+$. In fact, we must have $\pi_s(\pi_+) < \pi_n(\pi_+) \leq \pi_+$. (This is because if $\pi_s = \pi_n$, we must have $y = 0$, but this implies that $\pi = \delta\pi_s < \pi_s$.) Now moving π to the left. Note that for $\pi \geq \pi'$, we can obtain $u(\pi)$ by keeping the effort level and having $\frac{d\pi_n}{d\pi} = \frac{d\pi_s}{d\pi} = \frac{1}{\delta}$. (And by the uniqueness of the maximizer, this is the unique set of choices.) Now at π', we have $\pi_s(\pi') = \underline{\pi}$, $e(\pi') = e(\pi_+)$, and $\pi_n < \pi_+$.

Now consider values in $[\pi' - \varepsilon, \pi']$ for some small $\varepsilon > 0$. For all $\pi \in [\pi' - \varepsilon, \pi']$, note the following. Specifically, $u'(\pi) = u'(\pi_n) = u'_-(\pi_+)$, which is the slope of the line segment. (This is because for ε small enough, $\pi_n(\pi) < \pi_+$ by continuity of π_n). By L-e-n, this implies that effort in $[\pi' - \varepsilon, \pi']$ is constant.

Also note that all $\pi \in [\pi' - \varepsilon, \pi']$, we have $\pi_s(\pi) = \underline{\pi}$ by the monotonicity of π_s. This implies that $\pi = \delta\underline{\pi} + (1-\delta)y$. But if effort is constant, the above equality cannot hold for all $\pi \in [\pi' - \varepsilon, \pi']$. This leads to a contradiction. In other words, case 2 cannot happen. (In other words, if there is a line segment between $\underline{\pi}$ and π_+, we must have $\pi_n = \pi_s = \underline{\pi}$, and, thus, $\pi_s < \pi < \pi_n$.) ■

Lemma L4: *For all $\pi < \overline{\pi}$, we have $\pi < \pi_n(\pi)$.*

Proof. Suppose otherwise. There exists a π such that $\pi_s \leq \pi_n \leq \pi$. Note that these two inequalities cannot both be equalities. This implies from L-s-n that $u'_+(\pi_s) = u'_+(\pi)$, so u is a line segment between π_s and π. Let π_- be (the principal's value of) the left-most point of this line segment. By the previous lemma, we must have $\pi_- > \underline{\pi}$. Let the slope of this line segment be s.

Now moving π towards π_-. Define π' be the (maximal) principal's value such that $\pi_s(\pi') = \pi_+$. Note that at π', we have $\pi_n(\pi') \leq \pi'$. (This is because if $\pi_s(\pi) > \pi_-$, we can obtain $u(\pi)$ by setting $e(\pi) = e(\pi_+)$ and having $\frac{d\pi_n}{d\pi} = \frac{d\pi_s}{d\pi} = \frac{1}{\delta}$.) Now consider $\pi \in [\pi', \pi' + \varepsilon]$. By the monotonicity of π_s, we have $\pi_s \geq \pi'$. Therefore, $u'(\pi_s) = s$. In addition, we cannot have π_n strictly exceed π_+. Because if this were to happen, it violates L-s-n.

First, note that we cannot have both $\pi_s(\pi) < \pi_-$ and $\pi_n(\pi) > \pi'$. For this to happen, it means that y must increase (since the distance between π_s and π_n has increased). But if y increases and $\pi_n > \pi'$, we would violate $u'_-(\pi) \leq m(1-\theta)(1 + u'_-(\pi_n)) - \frac{c'}{y'}$ in L-e-n (since compared to $\pi = \pi'$, the right hand side decreases strictly yet the left hand side is constant.)

Second, we cannot have $\pi_s < \pi_-$. Because if this were to happen, we have $u'_-(\pi_s) > s$. In addition, we must have $u'_-(\pi_n) > s$ by the first point. And this violates L-s-n.

Therefore, by the monotonicity of π_s, we must have $\pi_s = \pi_-$. This implies that effort goes down as π moves to the left since $\pi = \delta\pi_s + (1-\delta)y$. Moreover, this implies that π_n goes down, and, thus, $u'(\pi_n) = s$ for $\pi \in [\pi' - \varepsilon, \pi']$. But this violates L-e-n. ■

Lemma L5: *If u is a line segment in $[\pi_-, \overline{\pi}]$, then $\pi_s < \pi$ for all $\pi \in [\pi_-, \overline{\pi})$.*

Proof. This follows because $\overline{\pi}$ is in region 3. Since $u'_+(\pi_s(\overline{\pi})) > u'_-(\overline{\pi})$, we must have $\pi_s(\overline{\pi}) \leq \pi_-$. Moreover, effort in region 3 is constant (given that u is a line segment). This implies that as π moves to the left from $\overline{\pi}$, π_s strictly decreases. Therefore, for all $\pi \in [\pi_-, \overline{\pi})$, we have $\pi_s(\pi) < \pi_-$. This proves the claim. ■

Lemma L6: *For $\pi > \underline{\pi}$, we have $\pi_s < \pi$.*

Proof. Suppose otherwise. There exists a π such that $\pi \leq \pi_s \leq \pi_n$. Note that these two inequalities cannot both be equalities. This implies from L-s-n that $u'_-(\pi_n) = u'_-(\pi)$, so u is a line segment between π and π_n. Let π_+ be (the principal's value of) the right-most point of this line segment. By the previous lemma, we must have $\pi_+ < \overline{\pi}$. Let the slope of this line segment be s.

Now moving π towards π_+. Define π' be the (minimal) principal's value such that $\pi_n(\pi') = \pi_+$. Note that at π', we have $\pi_s(\pi') \geq \pi'$. (This is because if $\pi_n(\pi) > \pi_+$, we can obtain $u(\pi)$ by keeping the effort level and having $\frac{d\pi_n}{d\pi} = \frac{d\pi_s}{d\pi} = \frac{1}{\delta}$.) Now consider $\pi \in [\pi', \pi' + \varepsilon]$ for small enough ε. By continuity of π_s, we remain having $u'(\pi_s) = s$. This implies that we cannot have $\pi_n(\pi) > \pi_+$. Because this violates L-s-n. When $m = 1$, we know that π_n is strictly increasing in π, so the argument above shows that when $m = 1$, we cannot have $\pi_s < \pi$. For below, we assume that $m \neq 1$).

Since $\pi_n(\pi) < \pi_+$, this implies that there exists a π'' such that $y(\pi'') < y(\pi')$. (recall that $\pi_n = \frac{1}{\delta}(\pi + (1-\delta)(m)y)$, and here we use that $\pi_+ < \overline{\pi}$.) At π'', however, since both $u'(\pi'') = u'(\pi_s(\pi'')) = s$, the above violates the L-e-s constraint. ■

Proposition L2: *For all $\pi \in (\underline{\pi}, \overline{\pi})$,*

$$\pi_s < \pi < \pi_n.$$

If $\pi_0 > \underline{\pi}$, then $\pi_s(\pi_0) = \underline{\pi}$. In this case, the relationship terminates with probability 1.

Proof. $\pi_s < \pi < \pi_n$ follows directly from the Lemma L4 and L6. Now consider $\pi_0 > \underline{\pi}$. Suppose to the contrary $\pi_s(\pi_0) > \underline{\pi}$, then from L-s-n, we see that $u'(\pi_0) = u'(\pi_s) = u'(\pi_n)$, and, thus, there exists $\pi_+ \geq \pi_0$ such that u is a line segment in $[\underline{\pi}, \pi_+]$. Then using the same argument as in Lemma L6, we get a contradiction.

To see that the relationship terminates with probability, it suffices to show that the continuation payoff falls below π_0 after finite number of shocks. Suppose the contrary. Then there exists a π such that $\pi_s^k(\pi) > \underline{\pi}$ for all k, where $\pi_s^k(\pi)$ denotes the continuation payoff after k^{th} consecutive shocks. Note that $\{\pi_s^k(\pi)\}_{k=1}^{\infty}$ is a decreasing sequence by above, it has a limit point, which we denote as $\pi_s^{\infty}(\pi)$. Now by the continuity of π_s, we have $\pi_s(\pi_s^{\infty}(\pi)) = \pi_s(\lim_{k\to\infty}(\pi_s^k(\pi))) = \lim_{k\to\infty}(\pi_s^{k+1}(\pi)) = \pi_s^{\infty}(\pi)$. But this is a contradiction because by the Lemma L6, we that for all $\pi > \underline{\pi}$, we have $\pi_s(\underline{\pi}) < \pi$. ■

9.4.4 Existence of the Left and Middle Region

The existence of the left and middle region is equivalent to that the liquidity constraint binds for some payoff level of the firm. One necessary and sufficient condition for the left or the middle region to exist is as follows. Define $\overline{\pi}^u$ as the maximal equilibrium payoff of the firm when the liquidity constraint is absent.

Lemma L7: *The PPE payoff frontier contains more than the right region, i.e., ($\pi_r > \underline{\pi}$) if and only if the following Condition L holds:*

$$\delta\overline{\pi}^u > (1+m)\underline{\pi}. \tag{L}$$

Proof. It suffices to look for the condition on whether the liquidity constraint is violated at $\underline{\pi}$ for the unconstrained problem. Under the unconstrained problem, $\pi_n(\underline{\pi}) = \overline{\pi}^u$ and $\pi_s(\underline{\pi}) = \underline{\pi}$. In addition, $\mathrm{RC_S}$ states that

$$\underline{\pi} = \delta\pi_s(\underline{\pi}) + (1-\delta)y(e(\underline{\pi})).$$

This implies that $y(e(\underline{\pi})) = (1-\delta)\underline{\pi}$. Therefore, the liquidity constraint that $\delta\pi_n \le \pi + (1-\delta)my(e)$ is equivalent to

$$\delta\overline{\pi}^u \le (1+m)\underline{\pi}.$$

■

Finally, we include a sufficient condition that the middle region exists. The proof used here provides a new way of showing that the value function is differentiable and may be of independent interest.

Lemma L8: *If Condition (L) is satisfied, the middle region exists if* $(1+m)(1-\theta) \le 1$ and $(1+m)(1-\theta)\frac{1+m}{m}\theta < 1$.

Proof. Suppose the contrary. Let π_d be the payoff that divides the left and the right region. It is immediate that u is not differentiable at π_d with

$$\begin{aligned} u'_+(\pi_d) &= -\frac{c'}{y'}; \\ u'_-(\pi_d) &= (1+m)(1-\theta)(1+u'(\overline{\pi})) - \frac{c'}{y'}. \end{aligned}$$

Let $\Delta u'(\pi_d) = u'_+(\pi_d) - u'_-(\pi_d) > 0$. Then L-e-s implies that

$$\Delta u'(\pi_d) \leq \frac{(1+m)\theta}{m} \Delta u'(\pi_s(\pi_d)).$$

This implies that u is not differentiable at $\pi_s(\pi_d)$.

Note that by L-e-n, we have

$$\Delta u'(\pi_s(\pi_d)) \leq (1+m)(1-\theta)\Delta u'(\pi_n(\pi_s(\pi_d))).$$

This implies that u is not differentiable at $\pi_n(\pi_s(\pi_d))$.

Since u is is differentiable for all $\pi \in \cup(\pi_d, \overline{\pi}]$, the above implies that either $\pi_n(\pi_s(\pi_d)) = \pi_d$ or $\pi_n(\pi_s(\pi_d)) \in (\pi_s(\pi_d), \pi_d)$. In the later case, we can show, using the same argument as above, that either $\pi_n^2(\pi_s(\pi_d)) = \pi_d$ or $\pi_n^2(\pi_s(\pi_d)) \in (\pi_n(\pi_s(\pi_d)), \pi_d)$, where the superscript denotes that applying π_n twice. Continuing with the same argument, we can show eventually that

$$\pi_d = \pi_n^K(\pi_s(\pi_d))$$

for some K. In addition, for all $k \leq K$, we have

$$\Delta u'(\pi_n^k(\pi_s(\pi_d))) \leq (1+m)(1-\theta)\Delta u'(\pi_n^{k+1}(\pi_s(\pi_d)))$$

Linking this chain of inequalities, we get

$$\Delta u'(\pi_d) \leq \frac{(1+m)\theta}{m}(1+m)^K(1-\theta)^K \Delta u'(\pi_d).$$

This is a contradiction. ■

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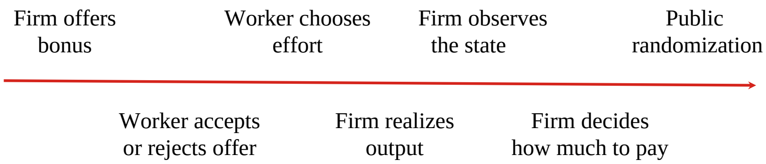
Figure 1



Stage Game Timing

Figure 2

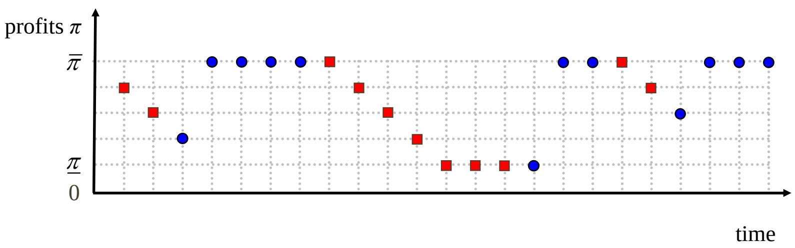


Figure 3

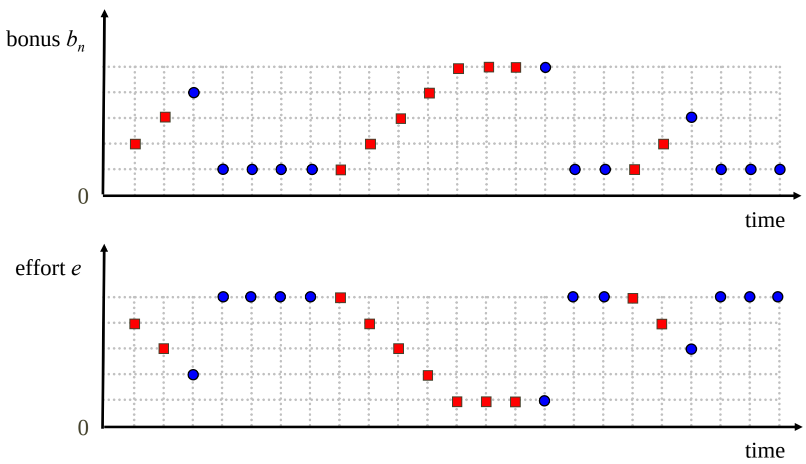


Figure 4